Control Systems

13

"Frequency domain analysis"

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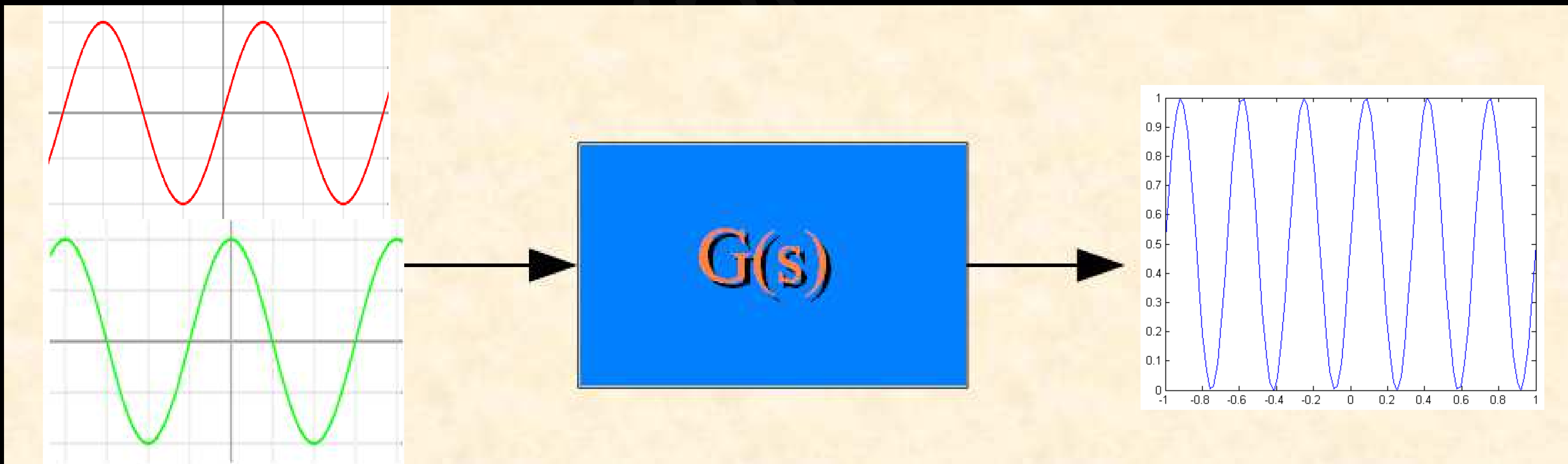
Frequency domain analysis

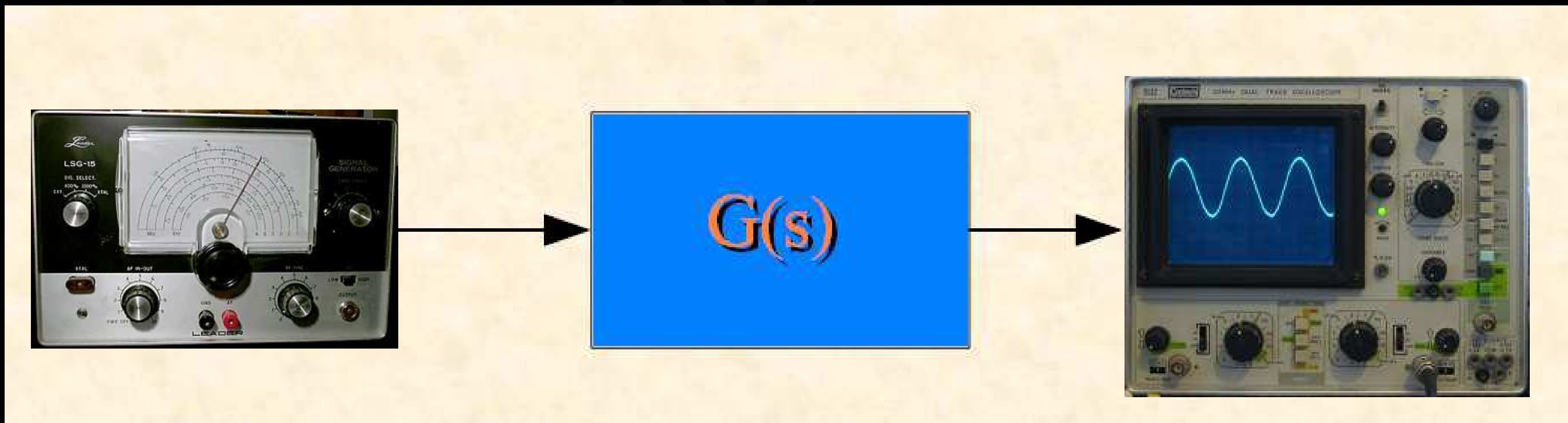
The frequency response is the output of the *system* in *steady state* when the input of the *system* is *sinusoidal*

Methods of *system analysis* by the frequency response, as the "Bode Diagrams" or the "Nyquist Plot", are the most conventional techniques used in Engineering for analysis and project of Control Systems



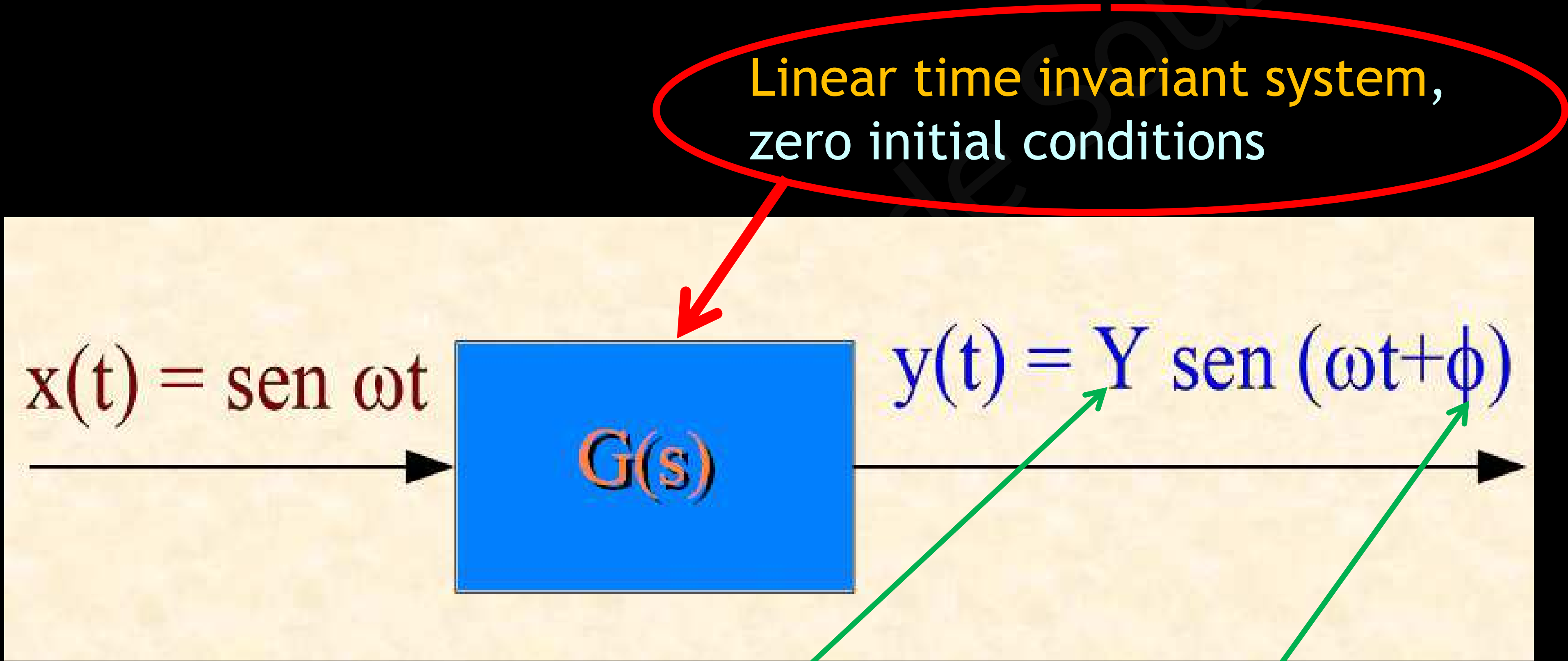
Frequency domain analysis

The advantage of these methods of analysis of systems by the frequency response is that they allow us to find both the *absolute* and *relative* stability of linear systems in *closed loop* only with the knowledge of frequency response in *open loop*, which can be experimentally obtained with signal generators (*sinusoidal*) and precision measurements instruments (*both easily available in laboratory*)



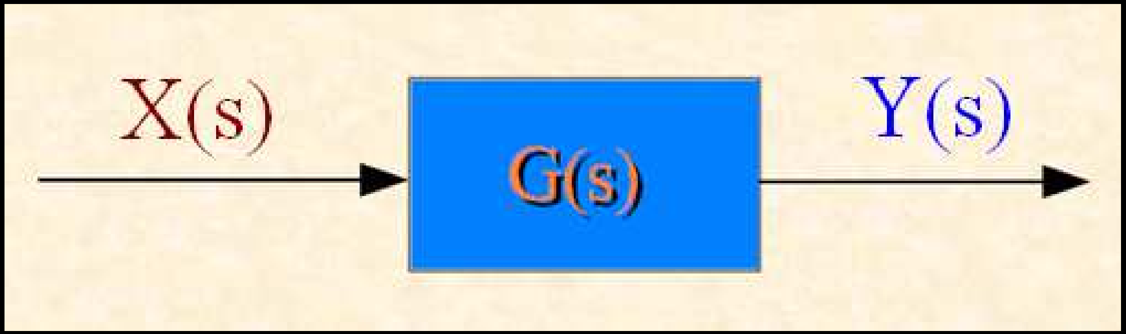
Frequency domain analysis

Therefore, the analysis of complicated systems can be done through tests of *frequency response* without being necessary to determine the *roots* of the characteristic equation (i.e., the *poles* of the *system*)

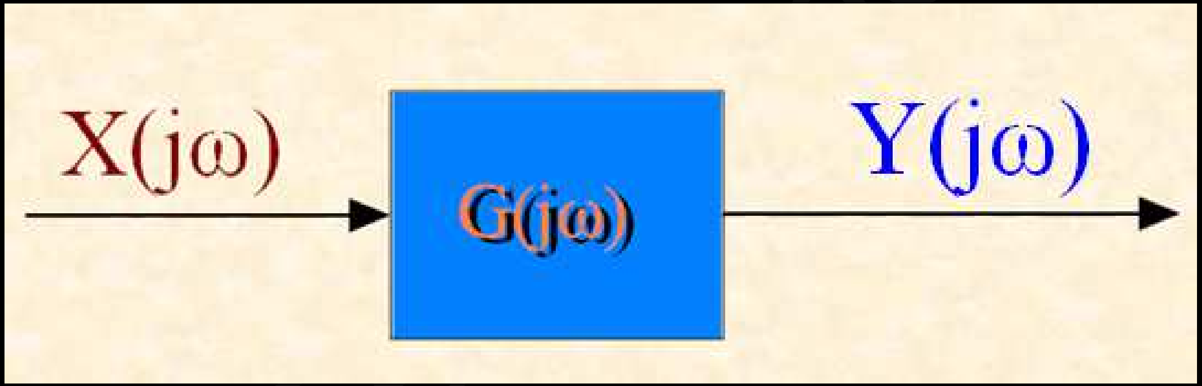


The output $y(t)$ will have the same *frequency* of the input $x(t)$, but, the amplitude Y and the phase angle ϕ, will be, in general, *different*.

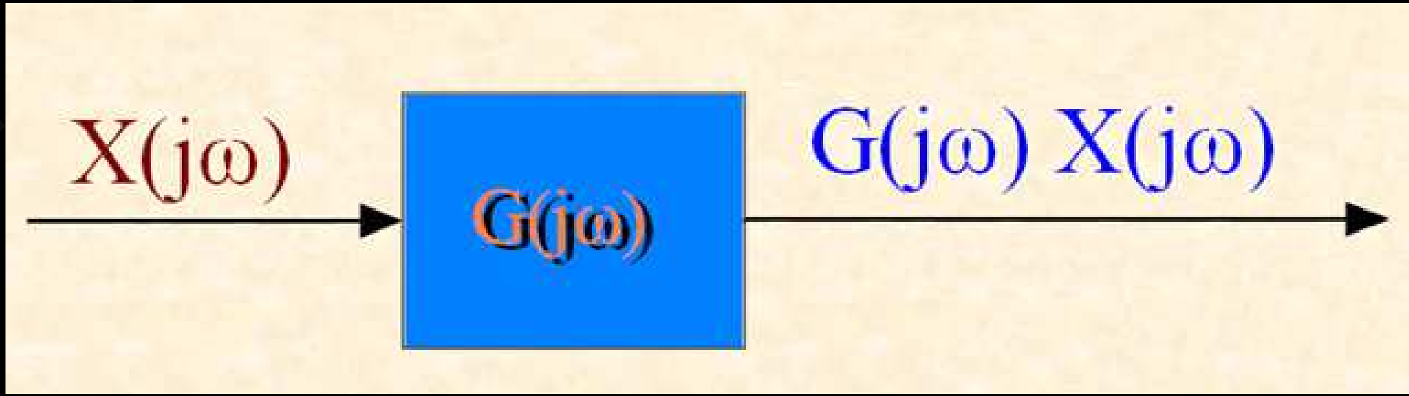
As a matter of fact, we have



And doing $s = 0 + j\omega$,



that is,



That is, doing $s = 0 + j\omega$ in the transfer function $G(s)$, one obtain $G(j\omega)$

$$G(j\omega) = |G(j\omega)| \cdot e^{j\phi}$$

$$= \underbrace{|G(j\omega)|}_{\text{Absolute value of } G(j\omega)} \cdot e^{j\angle G(j\omega)}$$

Absolute value of $G(j\omega)$

Phase of $G(j\omega)$

where the phase

$$\phi = \angle G(j\omega) = \text{arctg}\left(\frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))}\right)$$

As the *input* $x(t) = \sin(\omega t)$ can be expressed as

$$x(t) = \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$ (*equação de Euler*)

then it can be shown that y_{ss}, the *output* $y(t)$ in *steady state*, is

$$y_{ss} = |G(j\omega)| \cdot \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

and therefore,

$$y_{ss} = \underbrace{|G(j\omega)|}_{Y} \cdot \sin\left(\omega t + \underbrace{\angle G(j\omega)}_{\phi}\right)$$

(Absolute value) (Phase)

The most general case, the *sinusoidal input* with *absolute value* X and *phase* φ

$$x(t) = X \cdot \sin(\omega t + \varphi)$$

hence, y_{ss}, the *output* $y(t)$ in *steady state*, becomes

$$y_{ss} = \underbrace{X \cdot |G(j\omega)|}_{Y} \cdot \sin\Big(\omega t + \underbrace{\angle G(j\omega) + \varphi}_{\angle Y(j\omega)}\Big)$$

(Absolute value)

(Phase)

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|} = \frac{Y}{X}$$

ratio between the *output* and the *input*'s amplitude

$$\phi = \angle G(j\omega) = \angle Y(j\omega) - \angle X(j\omega)$$

difference between the phase angle of the *output* and the *input*

Then, the characteristic of a *system* subject to a *sinusoidal input* can be obtained directly from

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = |G(j\omega)| \cdot e^{-j\phi}$$

(F.T. *sinusoidal*)

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = |G(j\omega)| \cdot e^{-j\phi} \quad \left(\text{F.T. } \textit{sinusoidal} \right)$$

where the phase

$$\phi = \angle G(j\omega) = \begin{cases} < 0 & \text{phase delay} \\ = 0 & \text{in phase} \\ > 0 & \text{phase in advance} \end{cases}$$

being

$$-\pi < \phi < \pi$$

Se $\phi = \pi \Rightarrow \sin(\omega t \pm \pi) = -\sin(\omega t)$

In the next *2 slides* we shall see an EXAMPLE of the curves:

Bode Diagram de G(jω)

absolute value e phase (or angle)

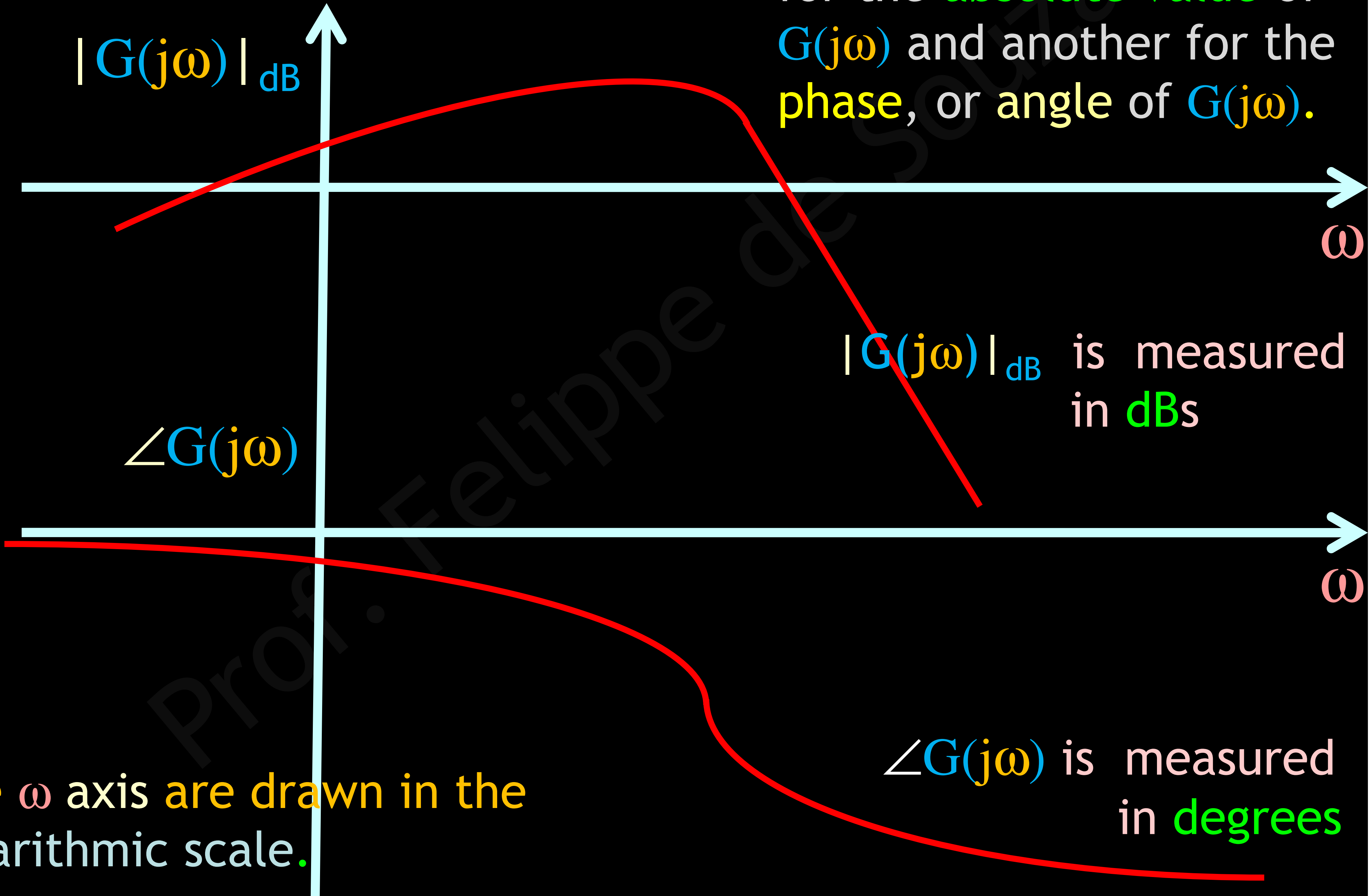
Nyquist Plot de G(jω)

parametrized by frequency ω

that is,
the next *2 slides* are generic curves to EXEMPLIFY how the Bode Diagrams and Nyquist Plots are.

Bode Diagram (example)

Obtained drawing a curve for the absolute value of $G(j\omega)$ and another for the phase, or angle of $G(j\omega)$.



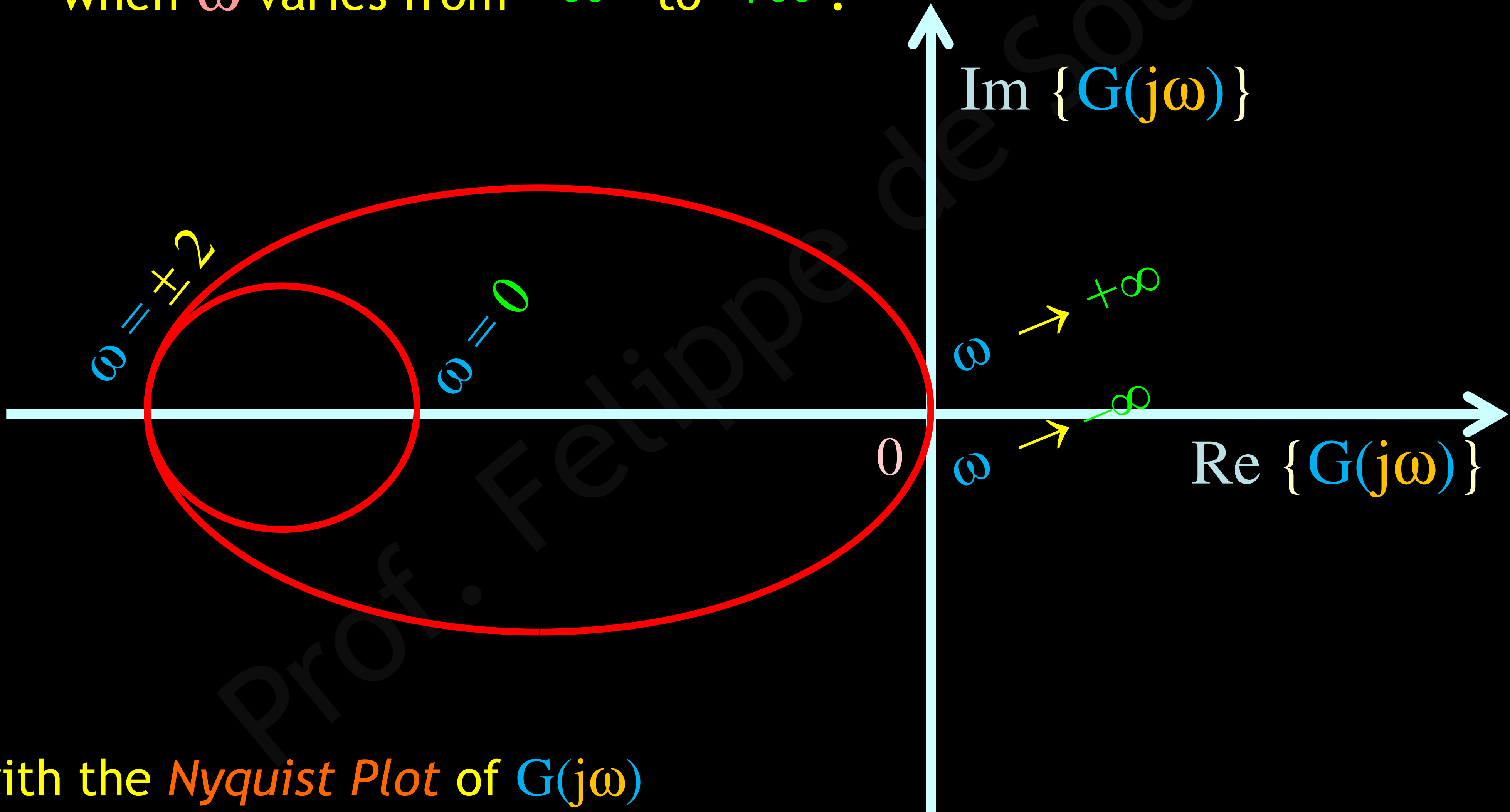
$|G(j\omega)|_{dB}$ is measured in dBs

$\angle G(j\omega)$ is measured in degrees

The ω axis are drawn in the logarithmic scale.

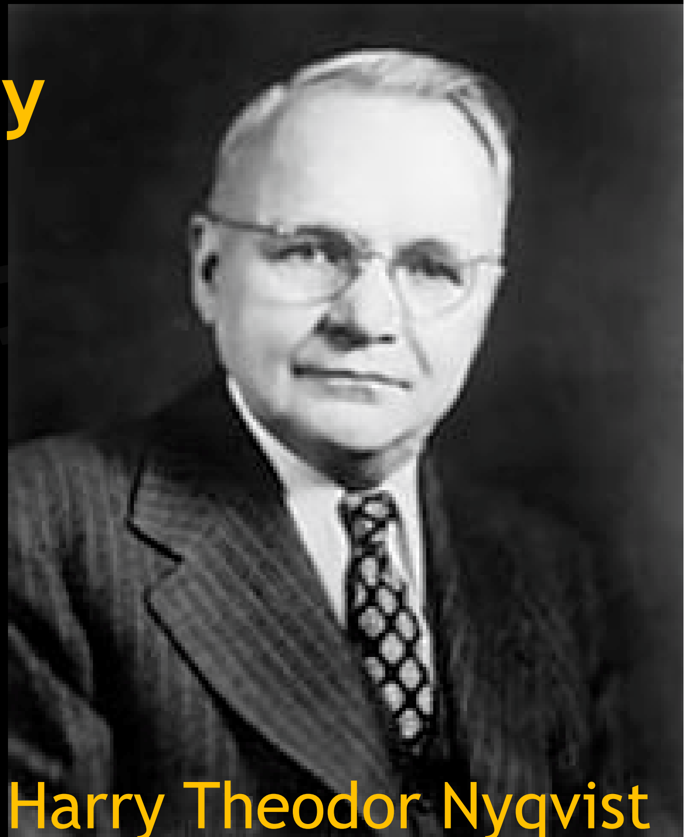
Nyquist Plot (example)

Obtained drawing in the complex plane the values of $G(j\omega)$ when ω varies from $-\infty$ to $+\infty$.



with the *Nyquist Plot* of $G(j\omega)$
we can apply the '*Nyquist Criterion*' to determine the stability

Nyquist Criterion for stability



Harry Theodor Nyqvist
(sueco, 1889-1976)

Harry Nyquist worked with the AT&T Company in 1917 and went on to produce 138 patents in the area of telephone and television transmission, as well as collecting many honours and awards.

Nyquist created the diagrams for defining stable conditions in negative feedback systems and the Nyquist sampling theory in digital communications.

The following is an Example of how to determine the direction and the number of encirclements of the Nyquist Plot

In the next *slides* we shall see two EXAMPLES of:

the direction and the number of encirclements of the

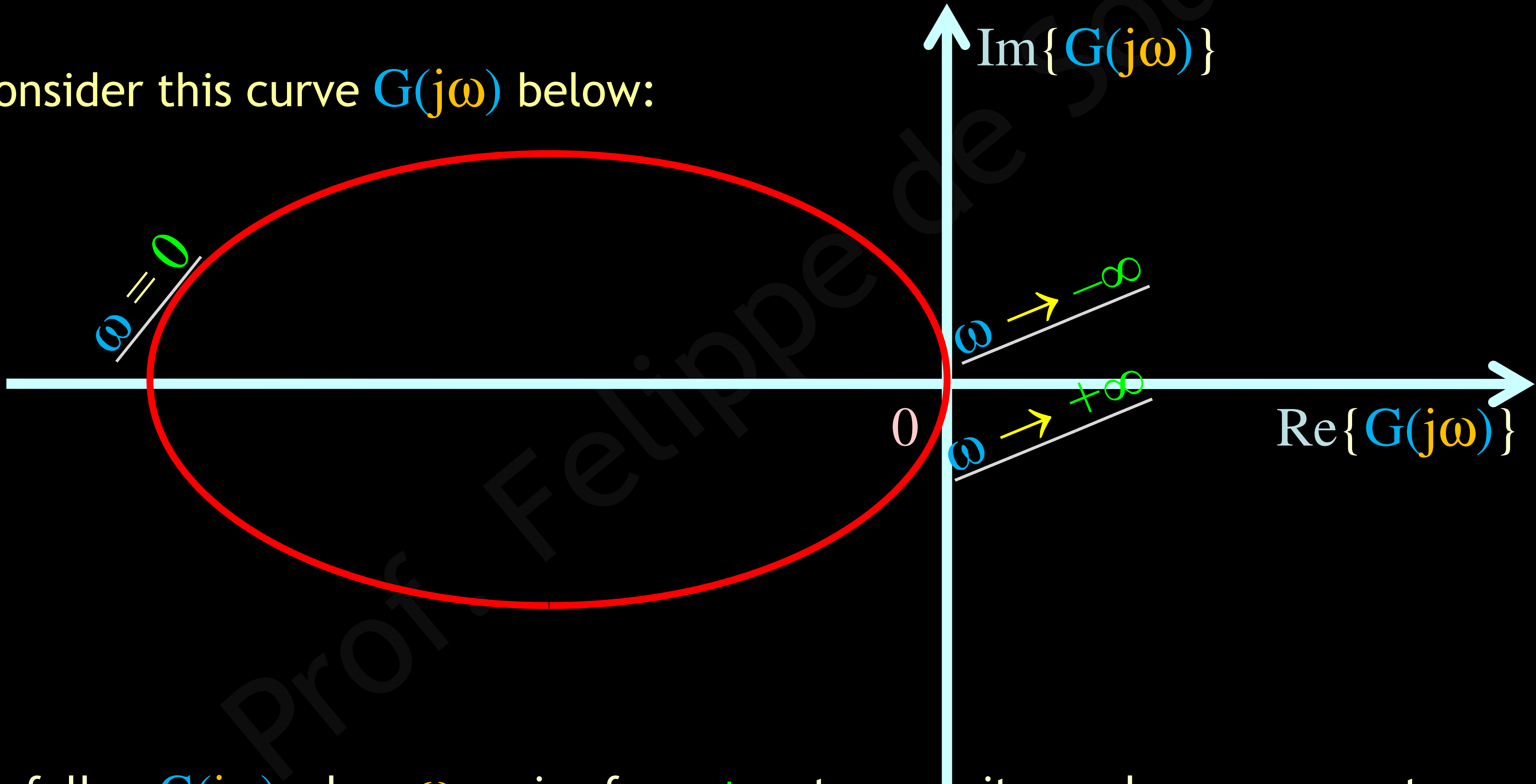
Nyquist Plot of $G(j\omega)$

We will use a generic curves only to EXEMPLIFY how to find the direction and how to calculate the number of encirclements of the Nyquist Plot.

The direction of the Nyquist Plot

In order to give a *direction* to the Nyquist Plot, we follow $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$

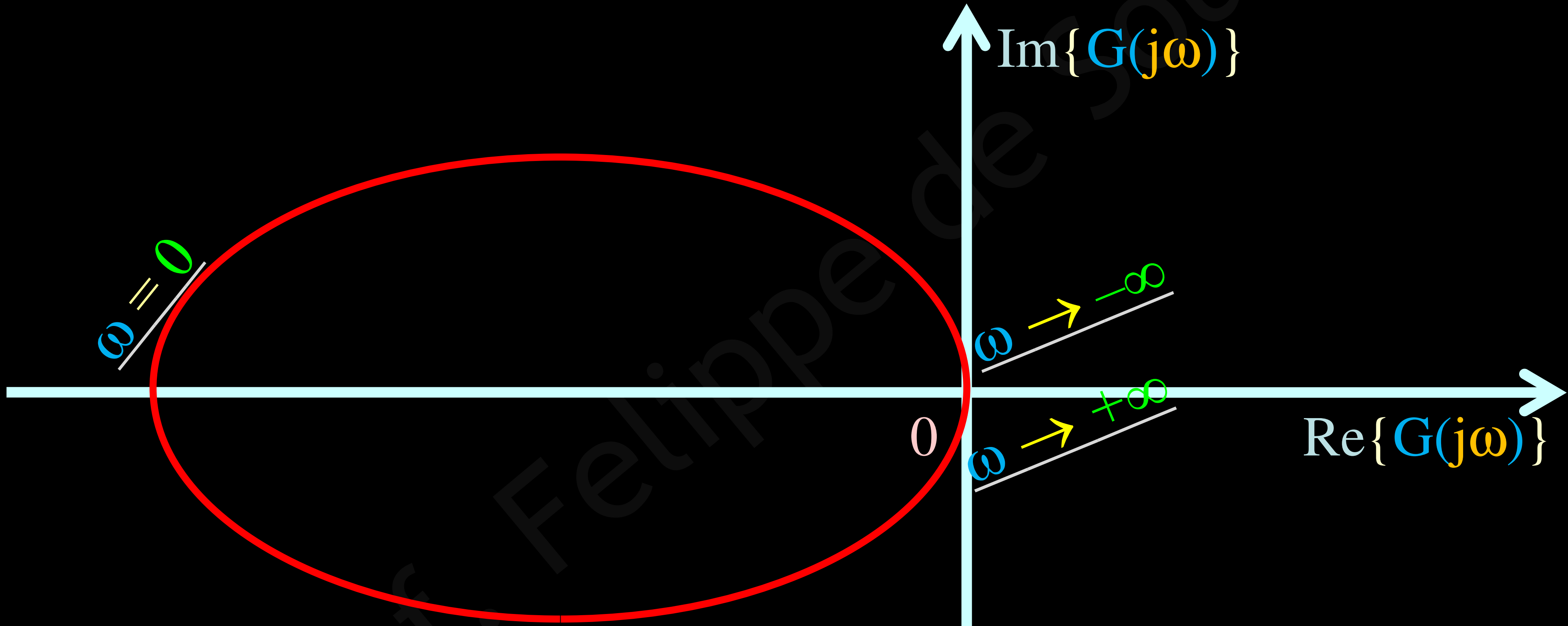
Consider this curve $G(j\omega)$ below:



To follow $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$, it may be necessary to calculate *additional points* of ω as in this Nyquist Plot above

The direction of the Nyquist Plot

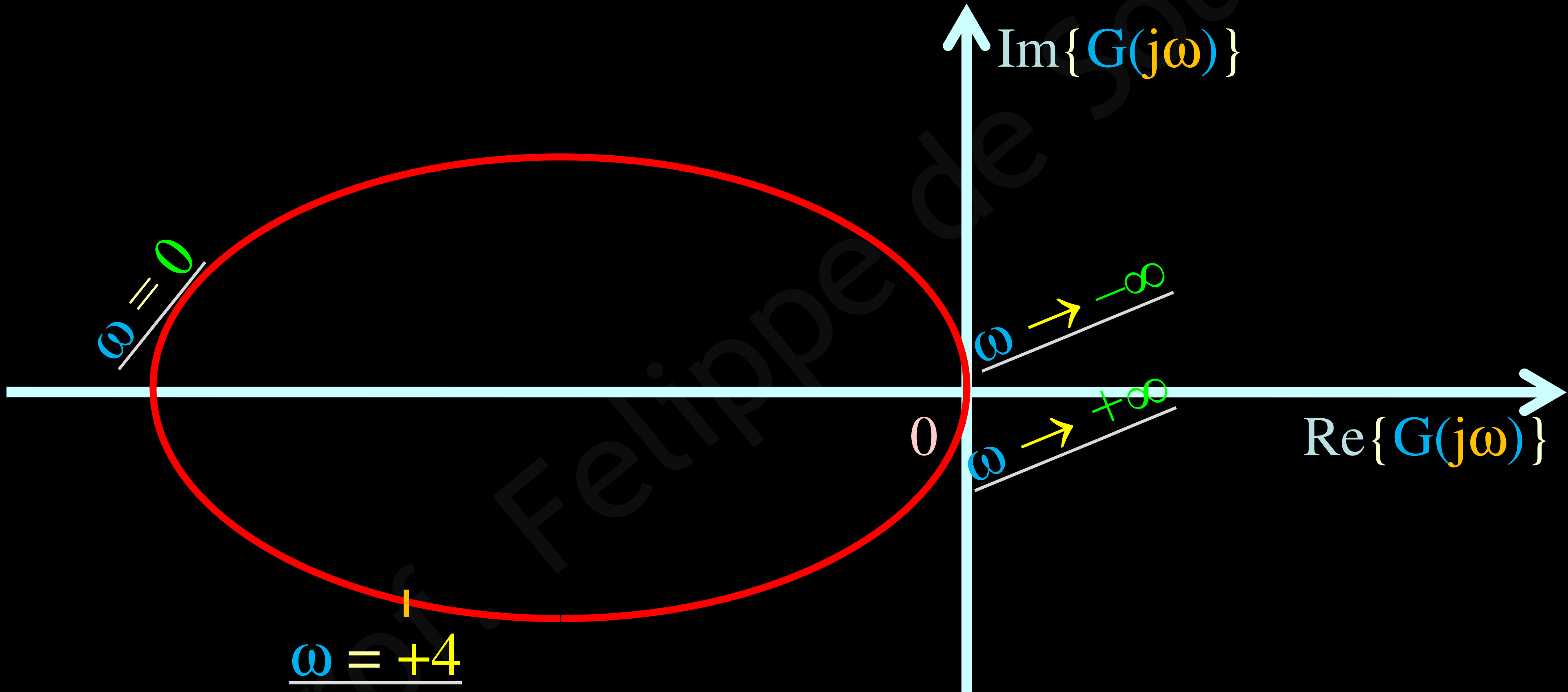
Following $G(j\omega)$ through the points $\omega = 0$ and $\pm\infty$ we can determine the *direction* of the Nyquist Plot



By calculating some additional points, we can then follow even better $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$ and then give a *direction* to this Nyquist Plot above

The direction of the Nyquist Plot

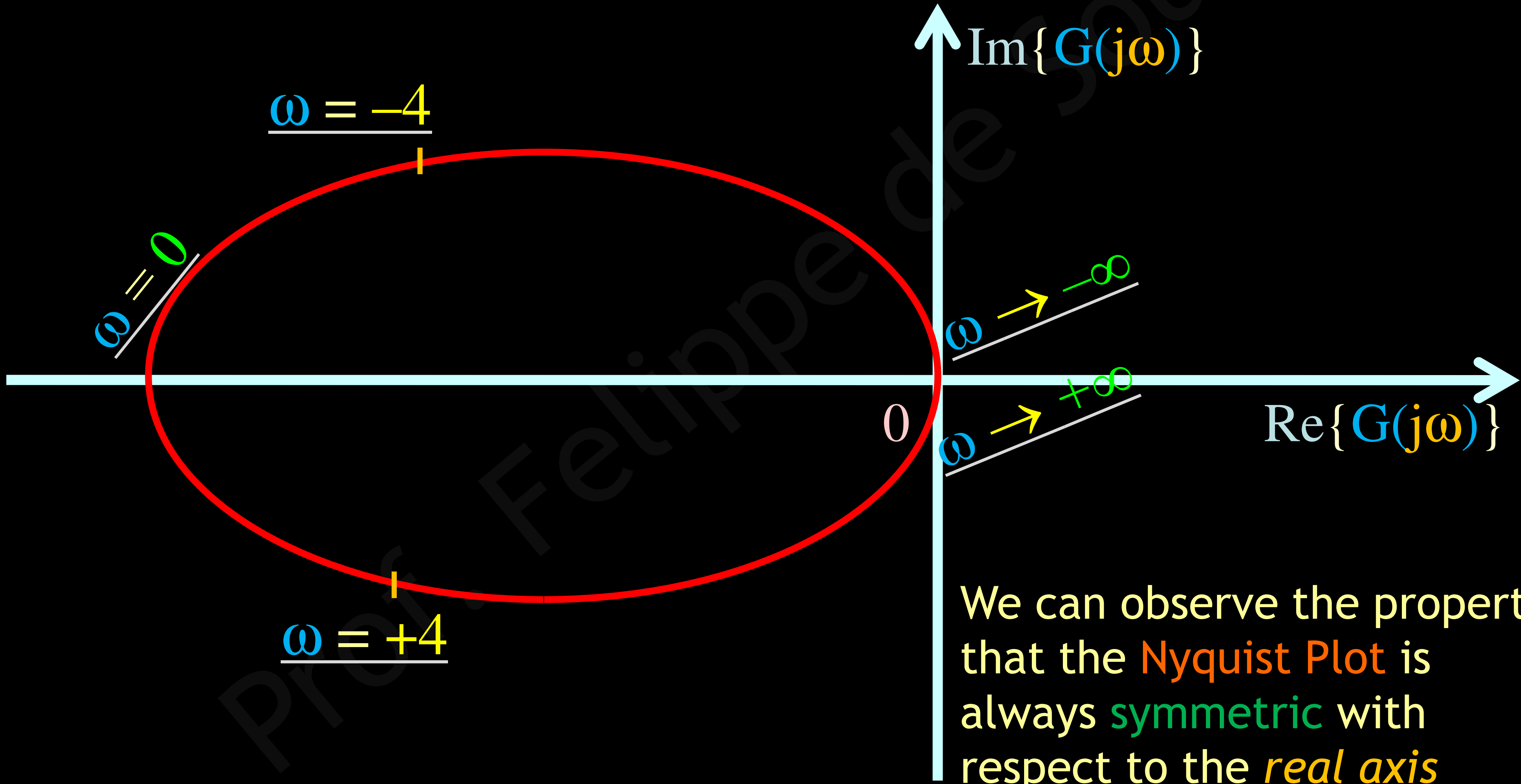
By choosing a ω between $+\infty$ and 0 (such as $\omega = 4$) it is possible to follow $G(j\omega)$ and determine the *direction* of this Nyquist Plot



Calculating $G(j\omega)$ for $\omega = 4$ we can write down this point $G(j4)$ in the Nyquist Plot above

The direction of the Nyquist Plot

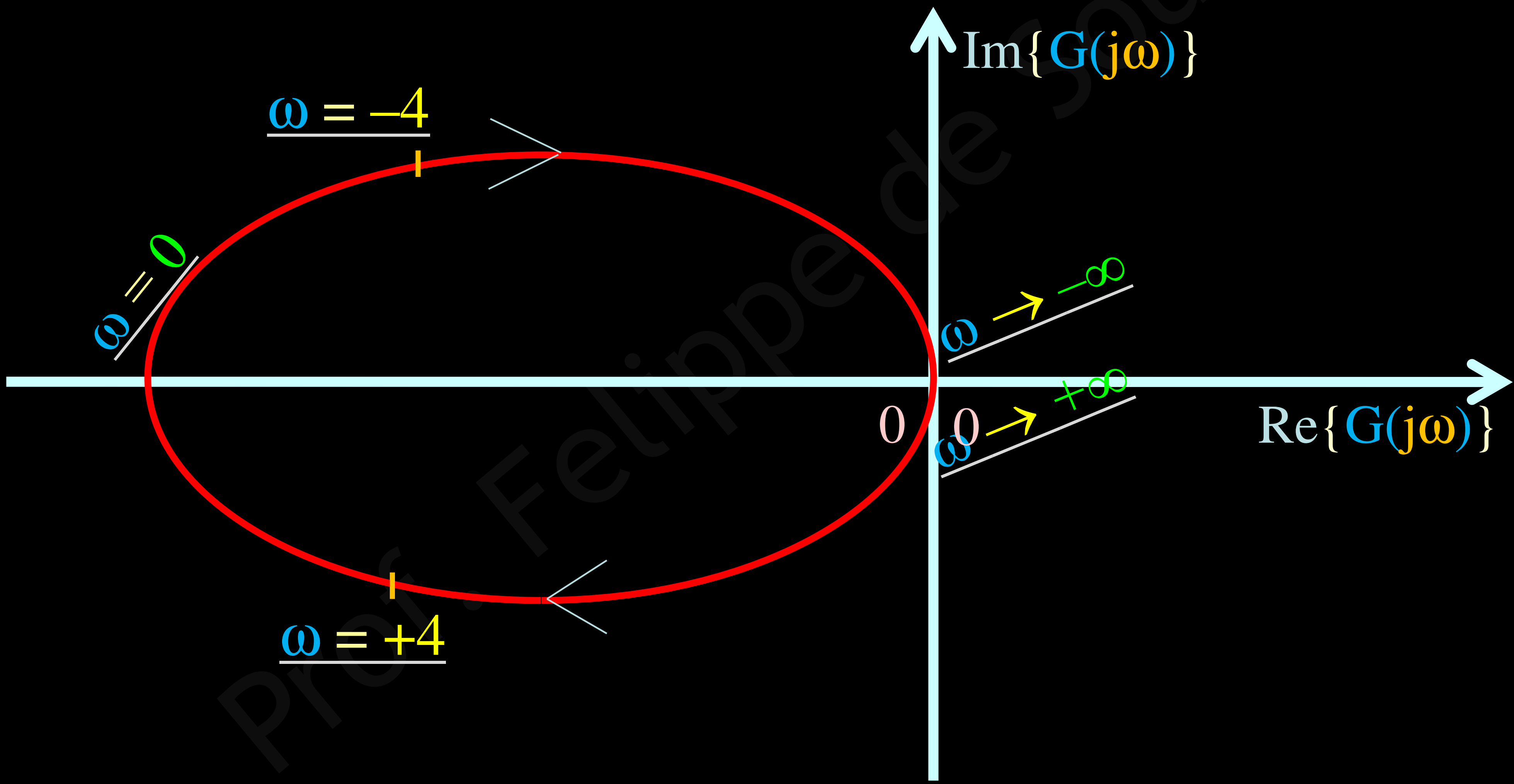
Although it is not necessary, let us also write down 1 point more in the plot $G(j\omega)$ below: $\omega = -4$



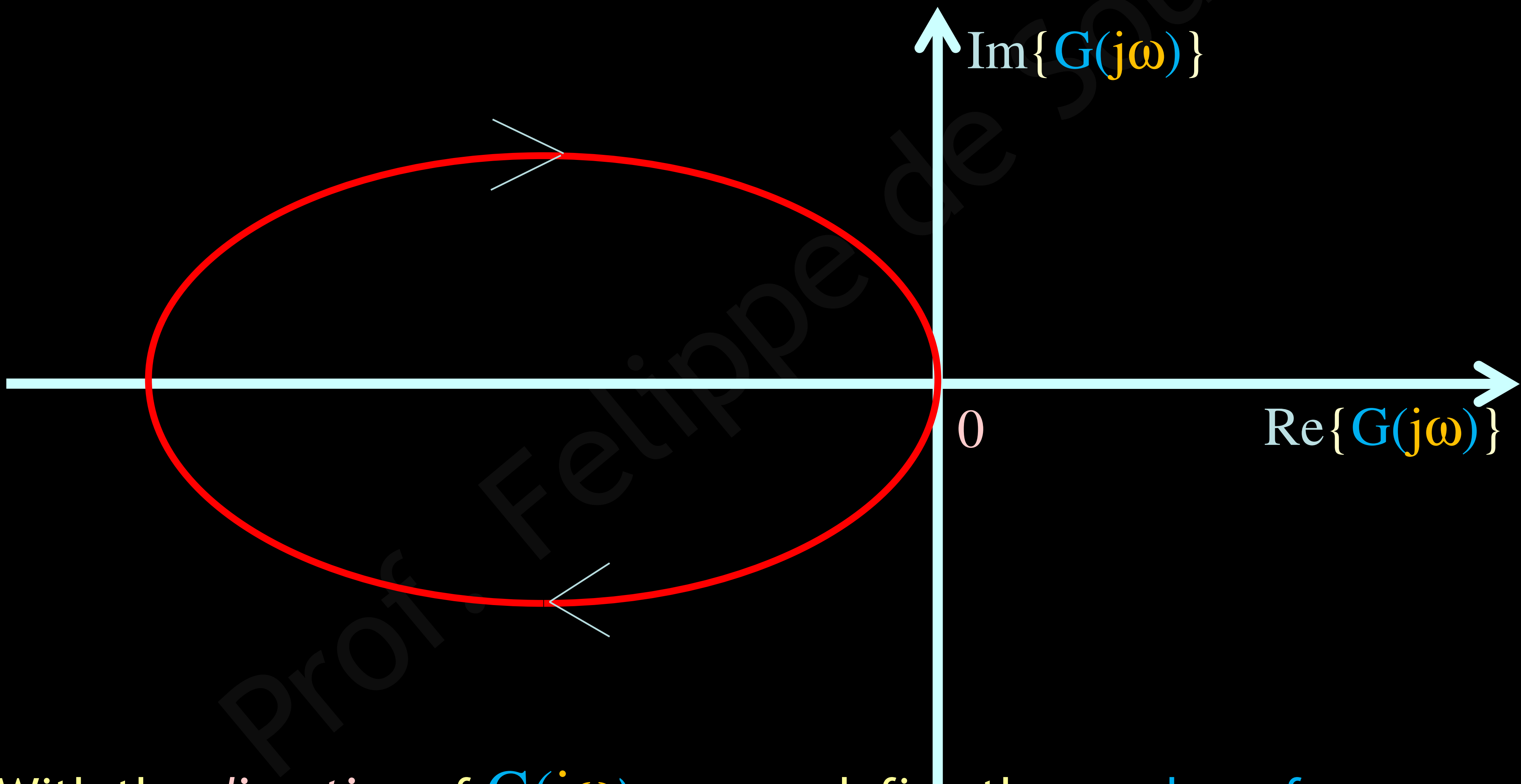
We can observe the property that the Nyquist Plot is always symmetric with respect to the real axis

The direction of the Nyquist Plot

Now we can put arrows that indicate the *direction* of this Nyquist Plot $G(j\omega)$



The direction of the Nyquist Plot

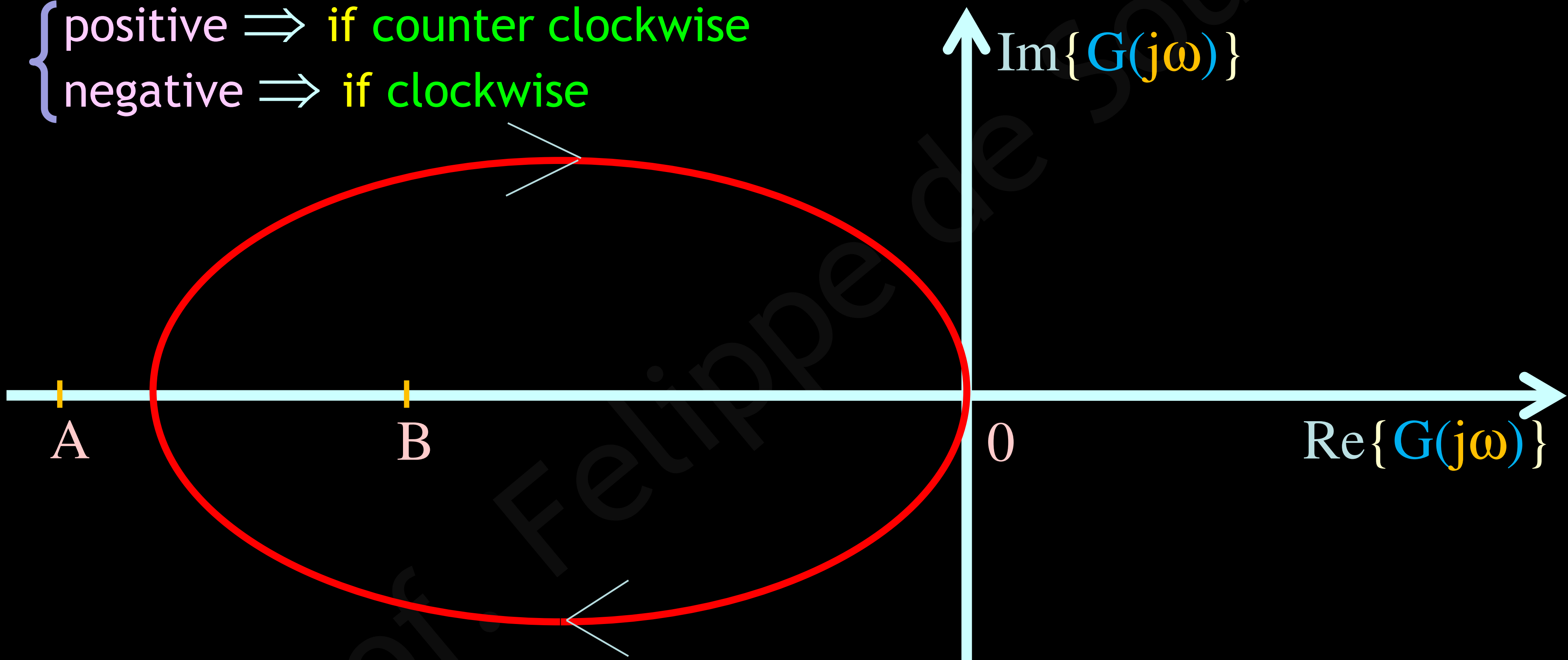


With the *direction* of $G(j\omega)$ we can define the number of encirclements of a point at *real axis* that the Nyquist Plot does

The number of encirclements of the Nyquist Plot

The number of encirclements of $G(j\omega)$ around a point at the *real axis* has the signal

- positive $\Rightarrow$ if counter clockwise
- negative $\Rightarrow$ if clockwise



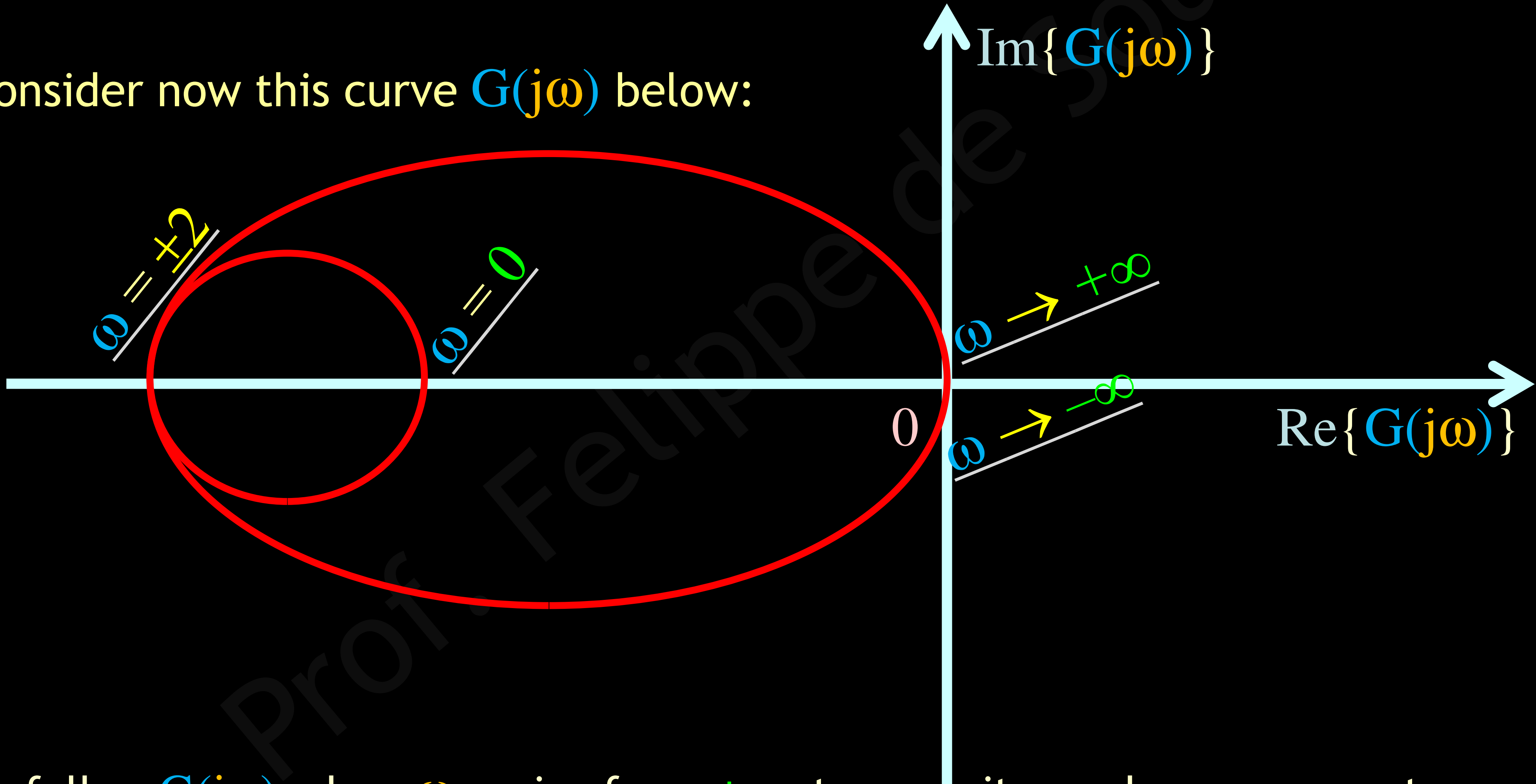
For example: The number of encirclements of $G(j\omega)$ around a *point*

- point A $\Rightarrow N_A = 0$
- point B $\Rightarrow N_B = -1$

The direction of the Nyquist Plot

Now let's see another example of how to give a *direction* to the Nyquist Plot, we follow $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$

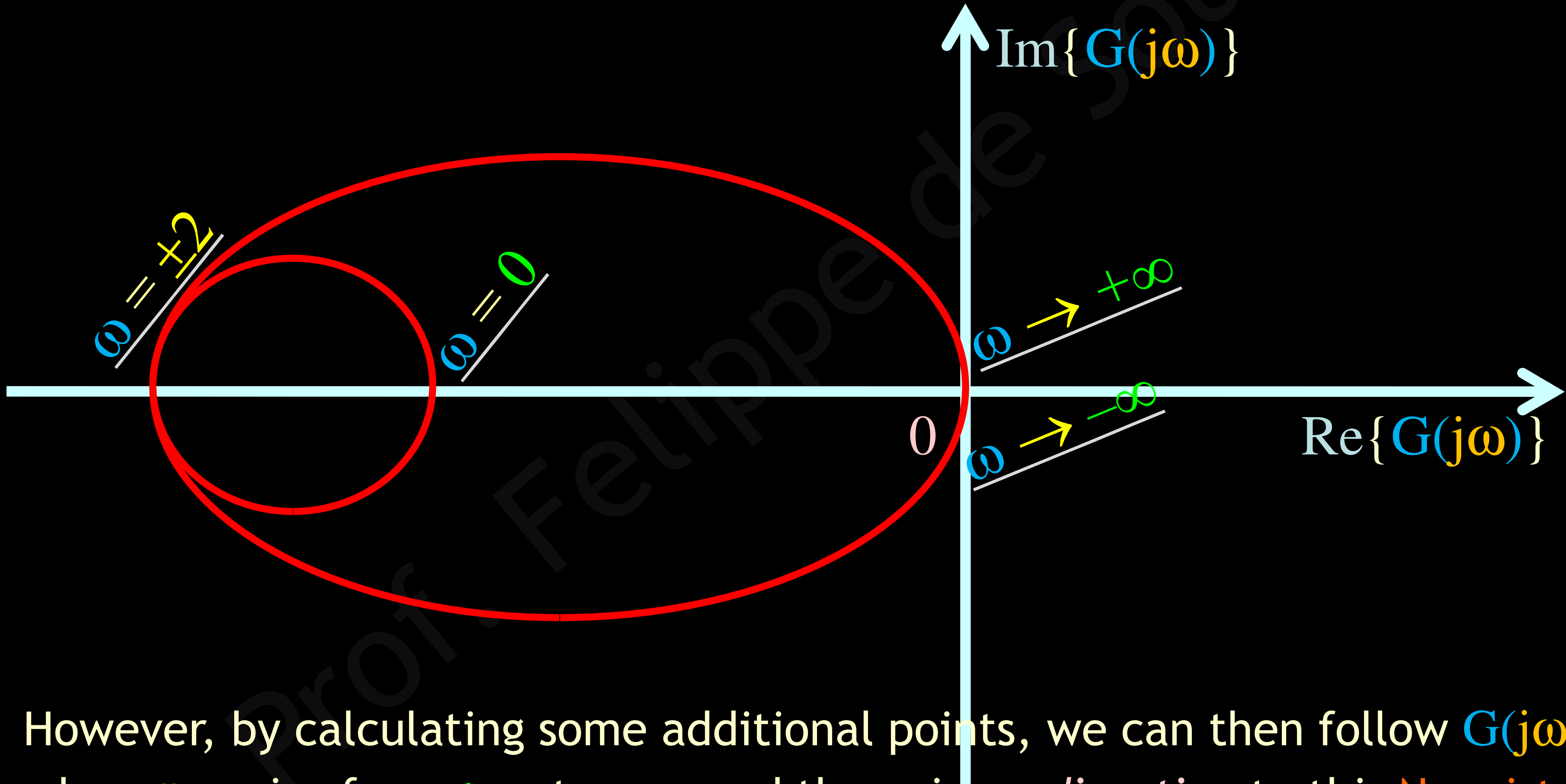
Consider now this curve $G(j\omega)$ below:



To follow $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$, it may be necessary to calculate *additional points* of ω as in this Nyquist Plot above

The direction of the Nyquist Plot

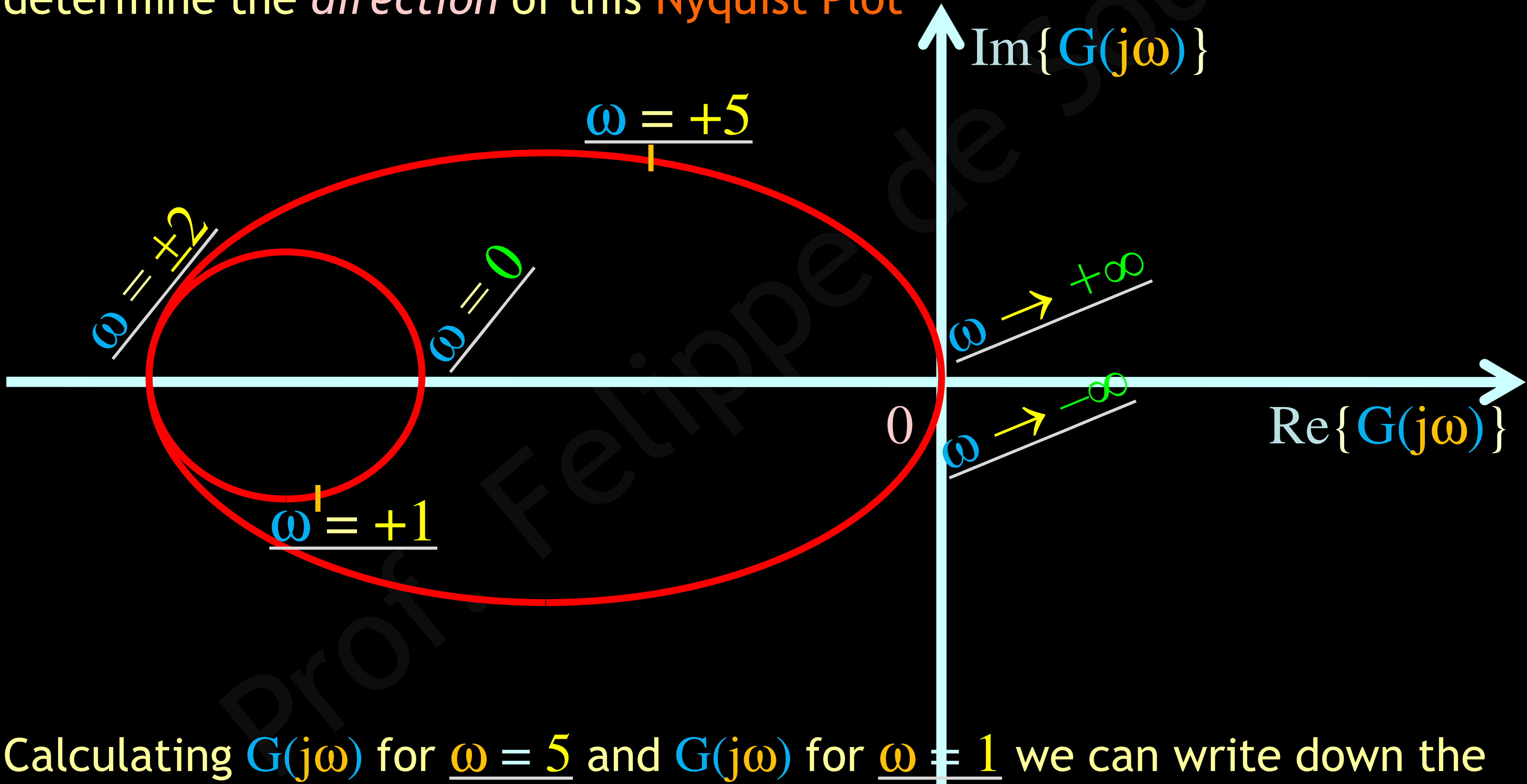
Following $G(j\omega)$ with only the points $\omega = 0, \pm 2$ and $\pm\infty$ it is not possible yet to determine the *direction* of the Nyquist Plot



However, by calculating some additional points, we can then follow $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$ and then give a *direction* to this Nyquist Plot above

The direction of the Nyquist Plot

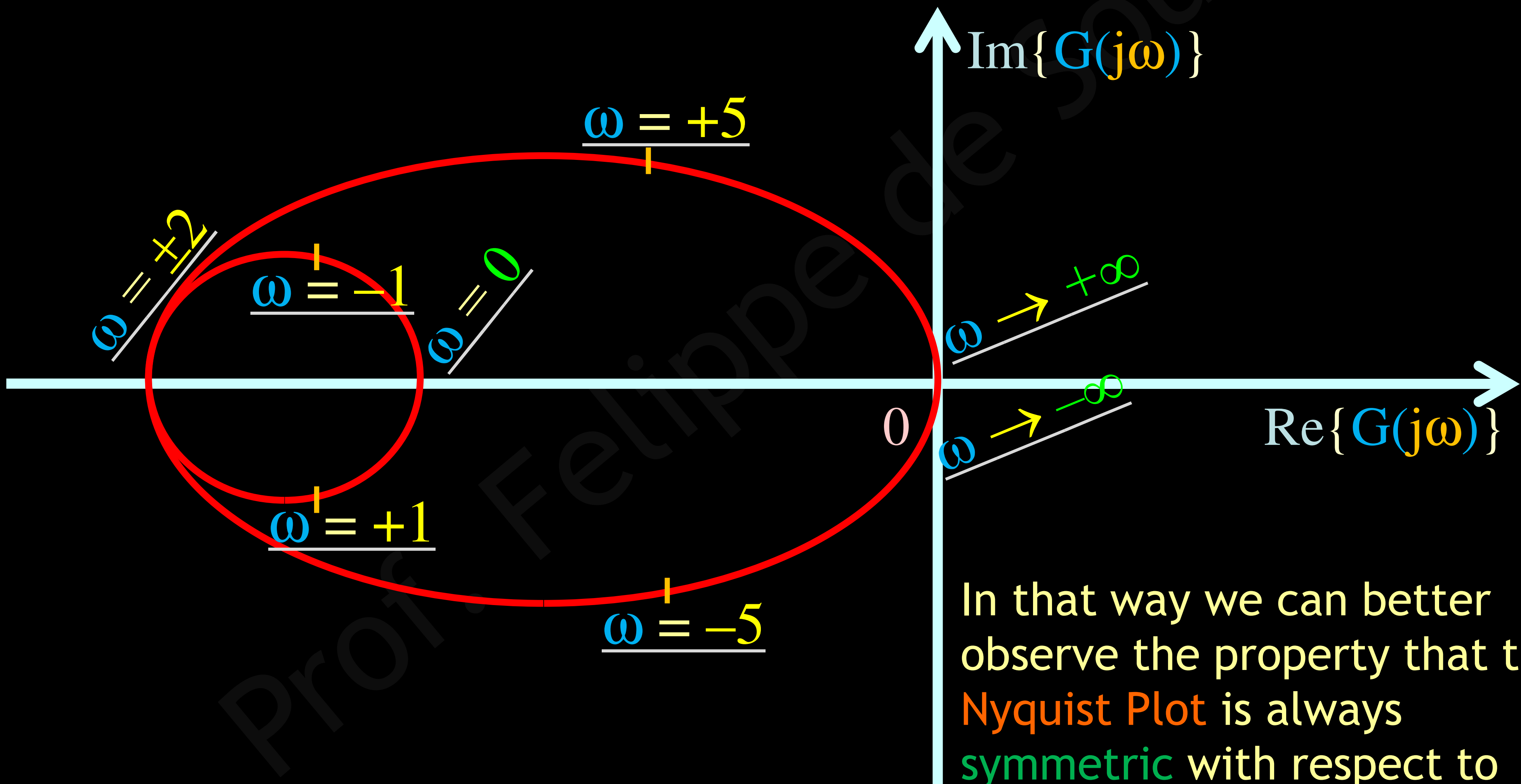
By choosing a ω between $+\infty$ and $+2$ (such as $\omega = 5$) and another ω between $+2$ and 0 (such as $\omega = 1$) it becomes possible to follow $G(j\omega)$ and determine the *direction* of this Nyquist Plot



Calculating $G(j\omega)$ for $\omega = 5$ and $G(j\omega)$ for $\omega = 1$ we can write down the points $G(j5)$ and $G(j1)$ in the Nyquist Plot above

The direction of the Nyquist Plot

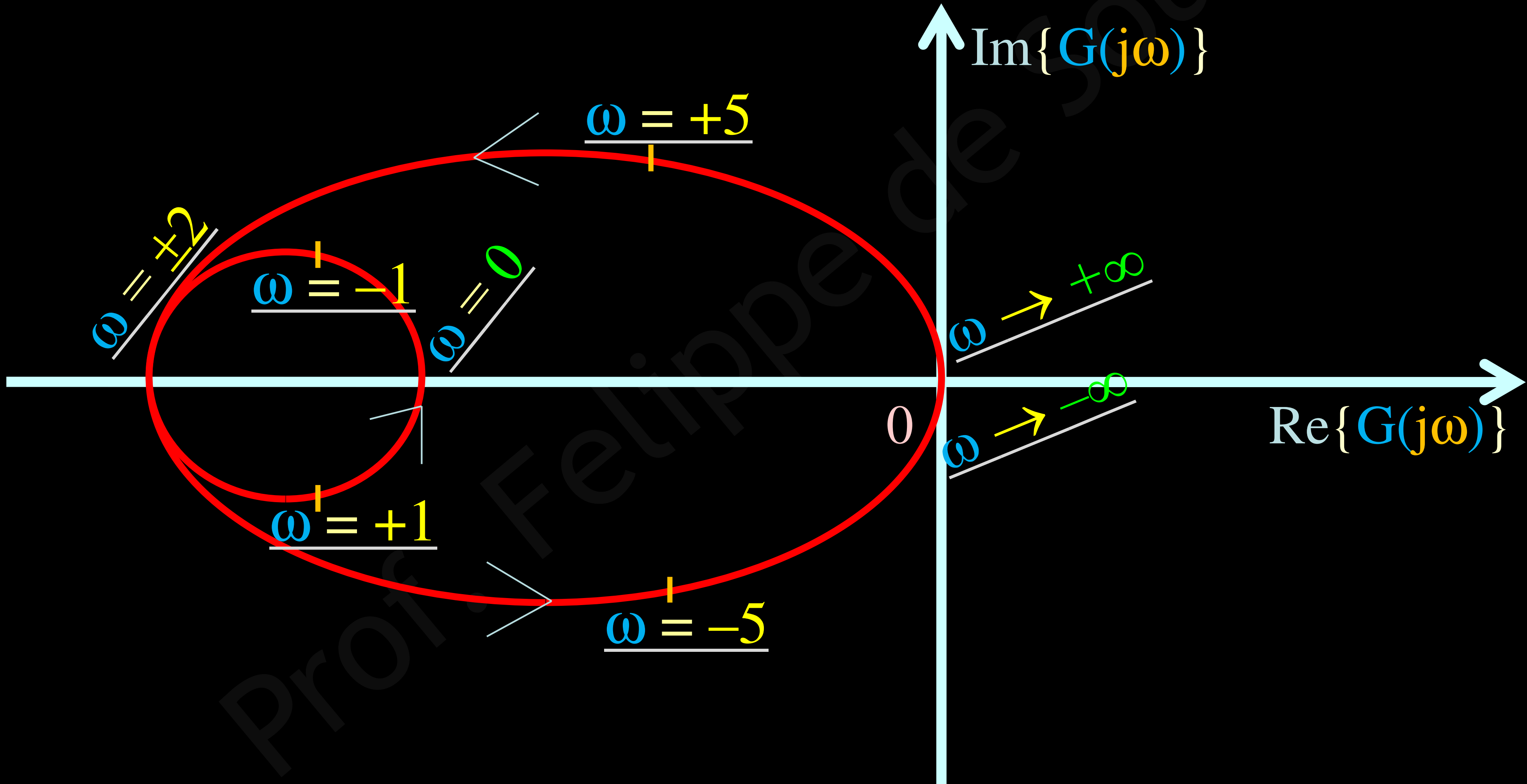
Although it is not necessary, let us also write down 2 points in the plot $G(j\omega)$ below: $\omega = -1$ e $\omega = -5$



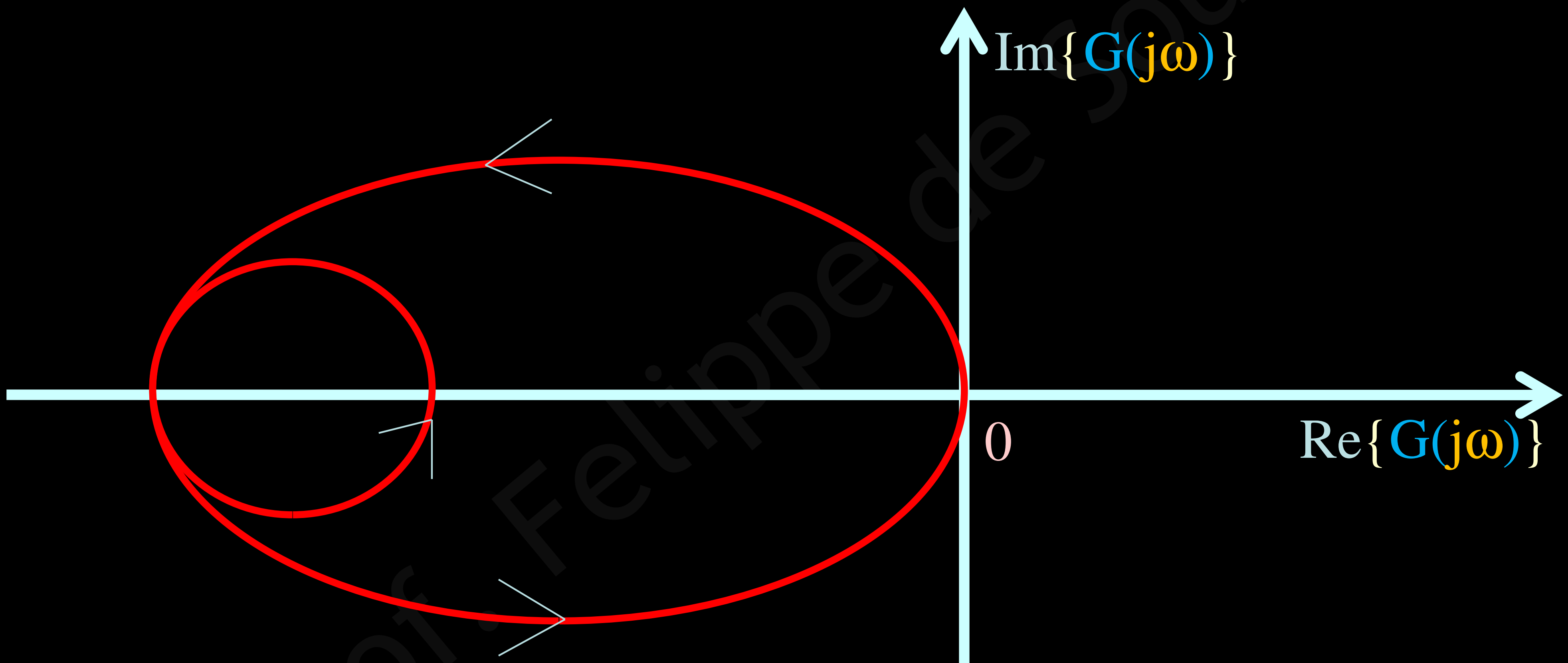
In that way we can better observe the property that the Nyquist Plot is always symmetric with respect to the *real axis*

The direction of the Nyquist Plot

Now we can put arrows that indicate the *direction* of this Nyquist Plot $G(j\omega)$



The direction of the Nyquist Plot

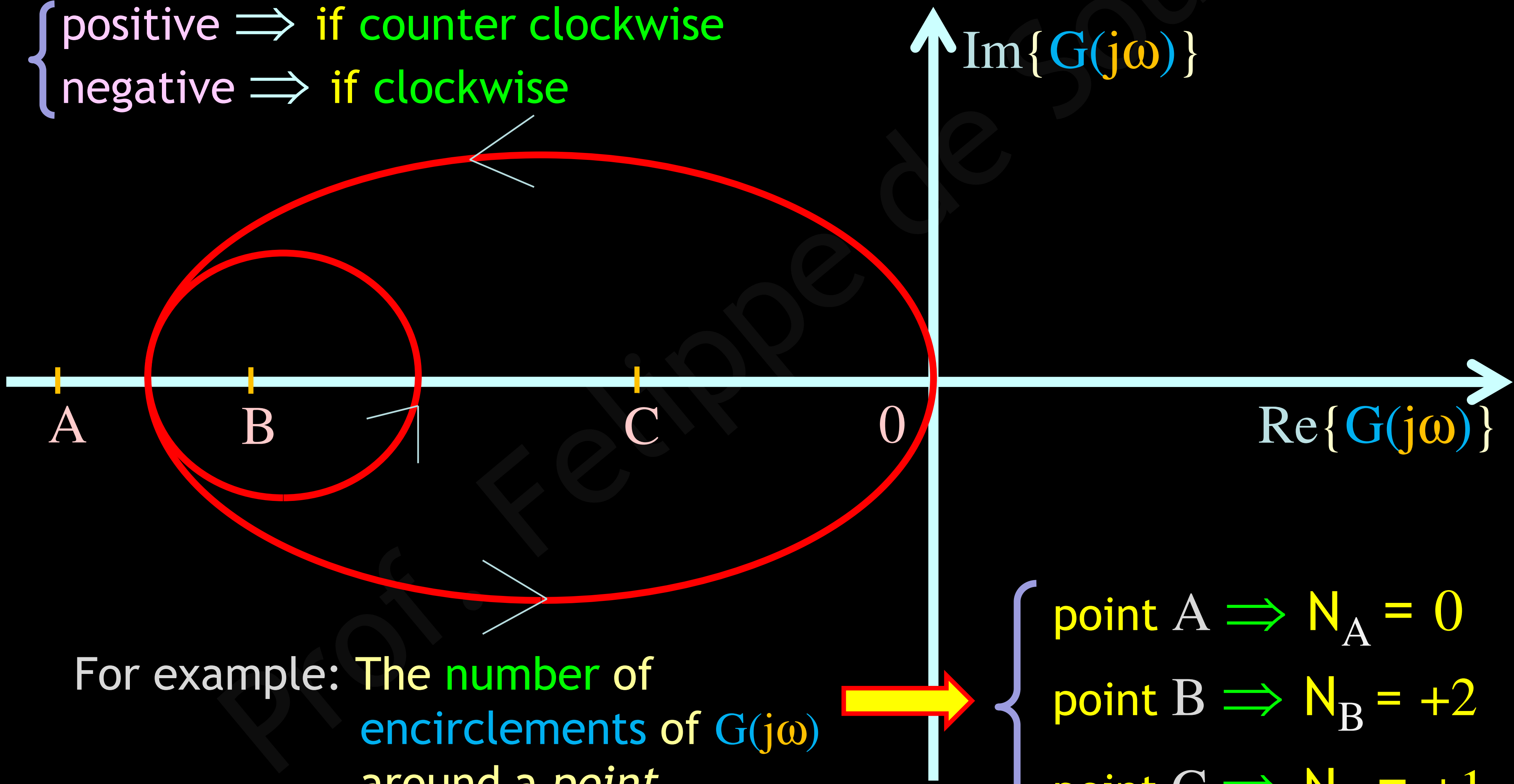


With the *direction* of $G(j\omega)$ we can define the number of encirclements of a point at *real axis* that the Nyquist Plot does

The number of encirclements of the Nyquist Plot

The number of encirclements of $G(j\omega)$ around a point at the *real axis* has the signal

- positive ⇒ if counter clockwise
- negative ⇒ if clockwise



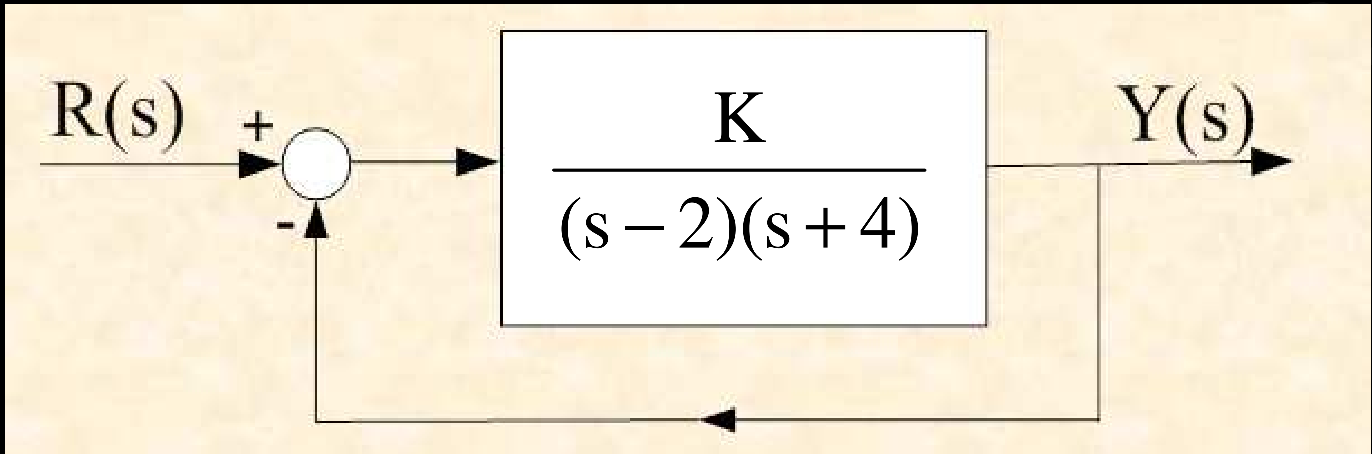
For example: The number of encirclements of $G(j\omega)$ around a *point*

- point A ⇒ $N_A = 0$
- point B ⇒ $N_B = +2$
- point C ⇒ $N_C = +1$

Next, we present some Example of

Nyquist Plot

Example 1: Consider the closed loop *system* below



thus,

$$G(s) = \frac{K}{(s-2)(s+4)}$$

and substituting $s = j\omega$, we obtain:

$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$

Example 1 (continued):

$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$

Intersection with the *real axis* (*imaginary part* = 0)

$$-2\omega = 0 \Rightarrow \omega = 0 \Rightarrow G(j\omega) = G(j0) = -K/8$$

Intersection with the *imaginary axis* (*real part* = 0)

$\nexists\,\omega$ real that makes null $\mathrm{Re}\{G(j\omega)\}$, the real part of $G(j\omega)$ $\Rightarrow$

➡ { this Nyquist Plot **does not** intercept the *imaginary axis*

Example 1 (continued):

$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$

Limits at infinite ($G(j\omega)$ for $\omega = \pm\infty$)

$$G(j\infty) = 0^- + j0^-$$

(3° quadrant)

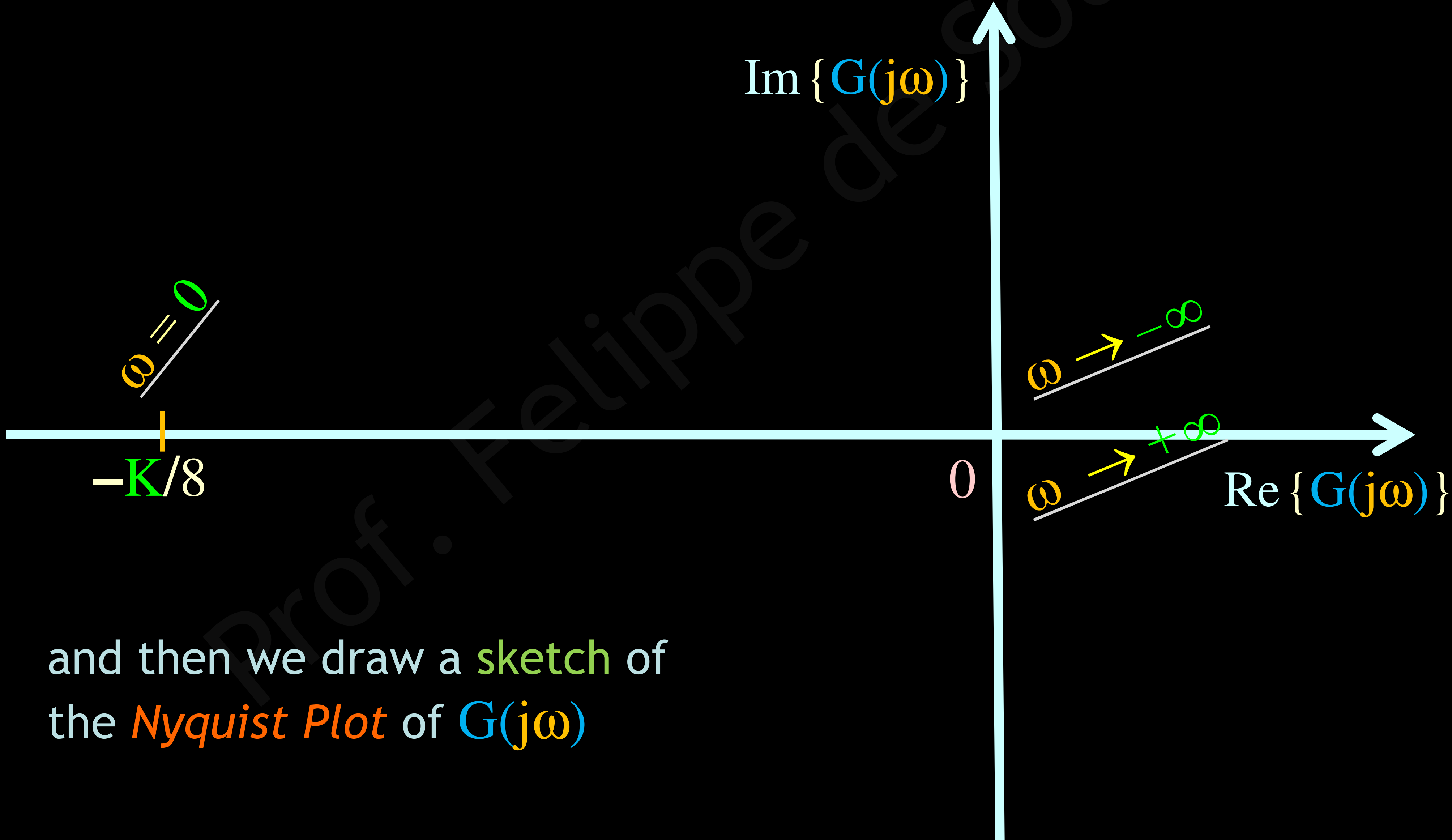
$$G(-j\infty) = 0^- + j0^+$$

(2° quadrant)

and now we write down in the complex plane
these *points of intersection* and the *limits at infinite*

Example 1 (continued):

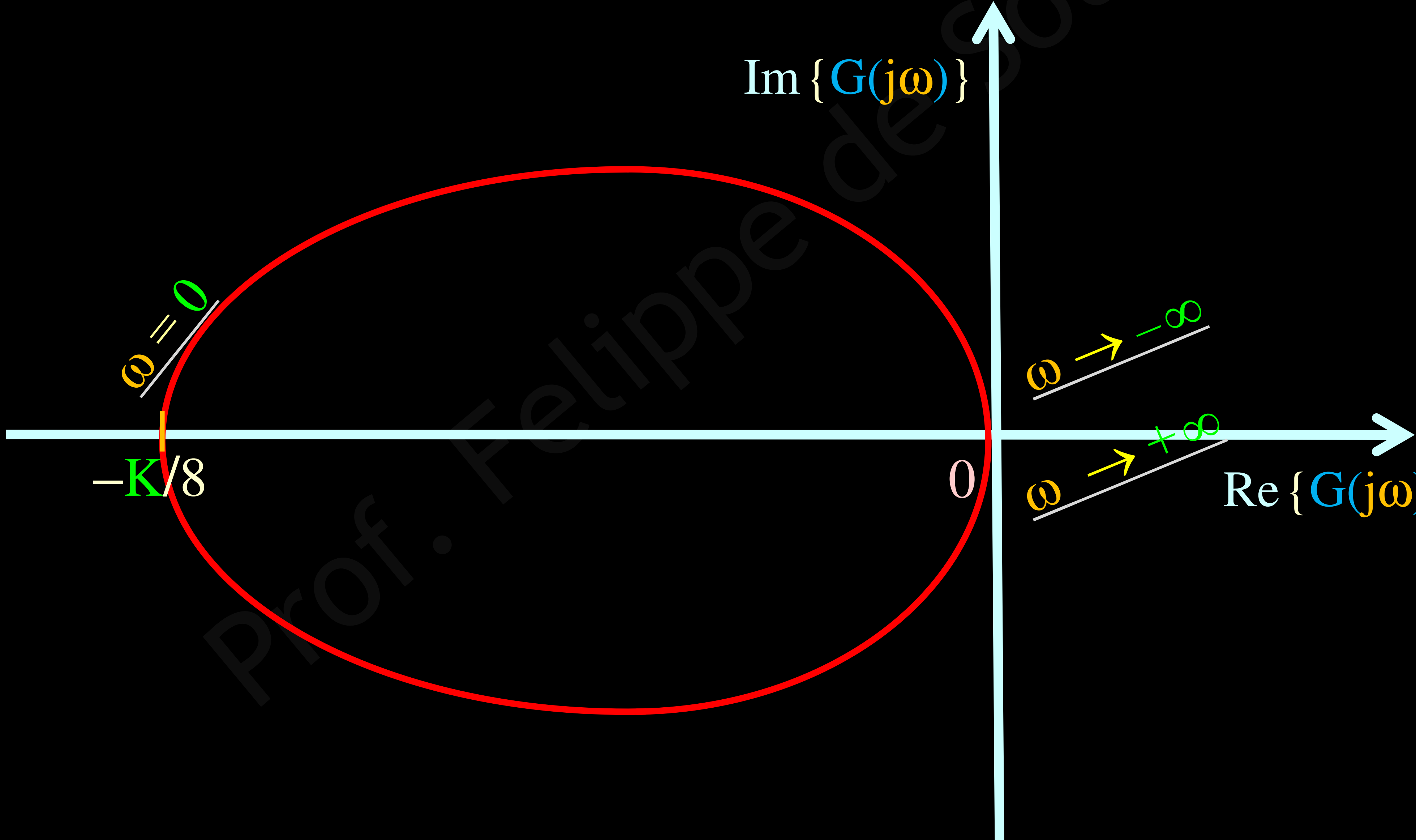
$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$



and then we draw a sketch of the *Nyquist Plot* of $G(j\omega)$

Example 1 (continued):

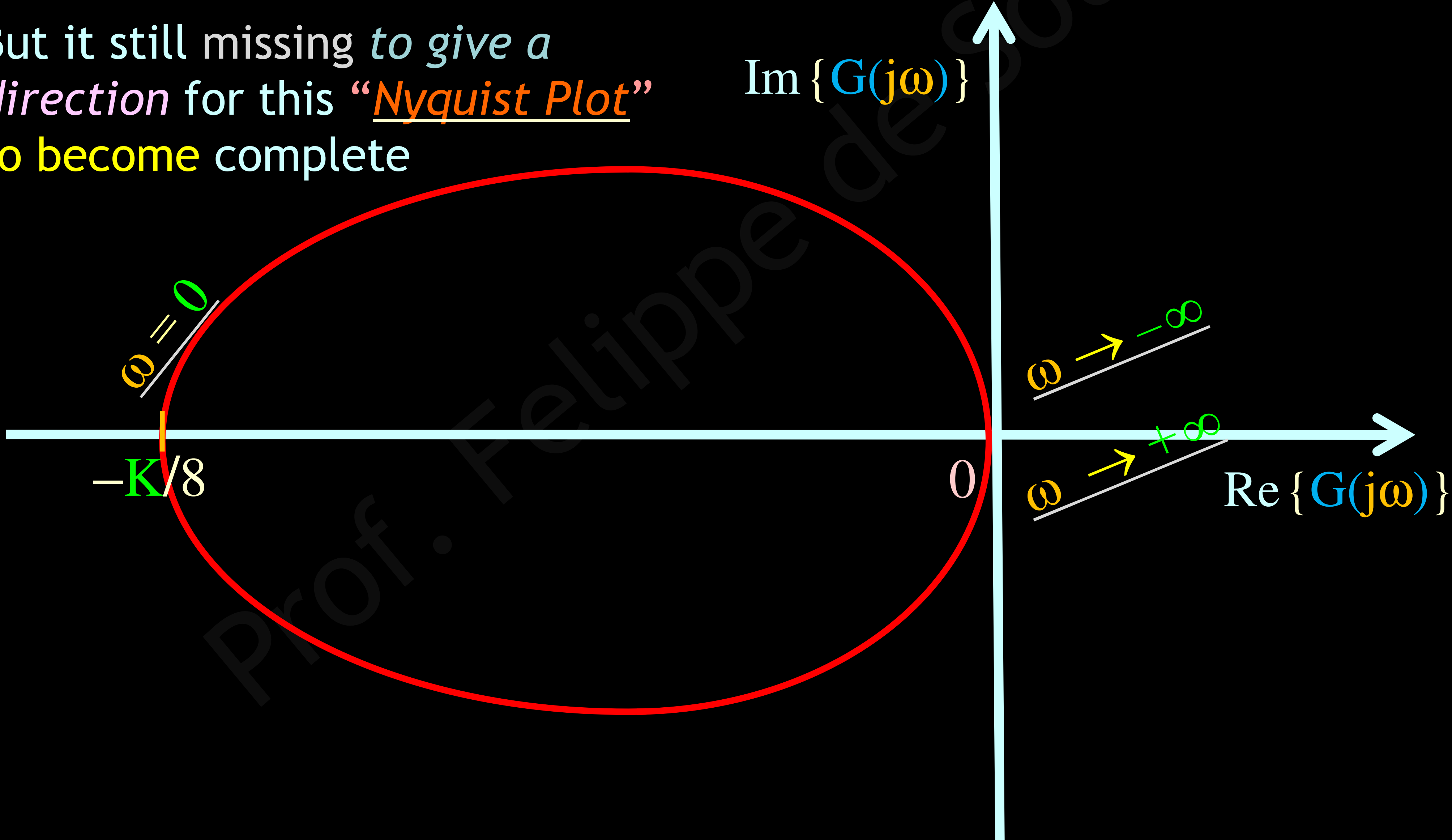
$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$



Example 1 (continued):

$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$

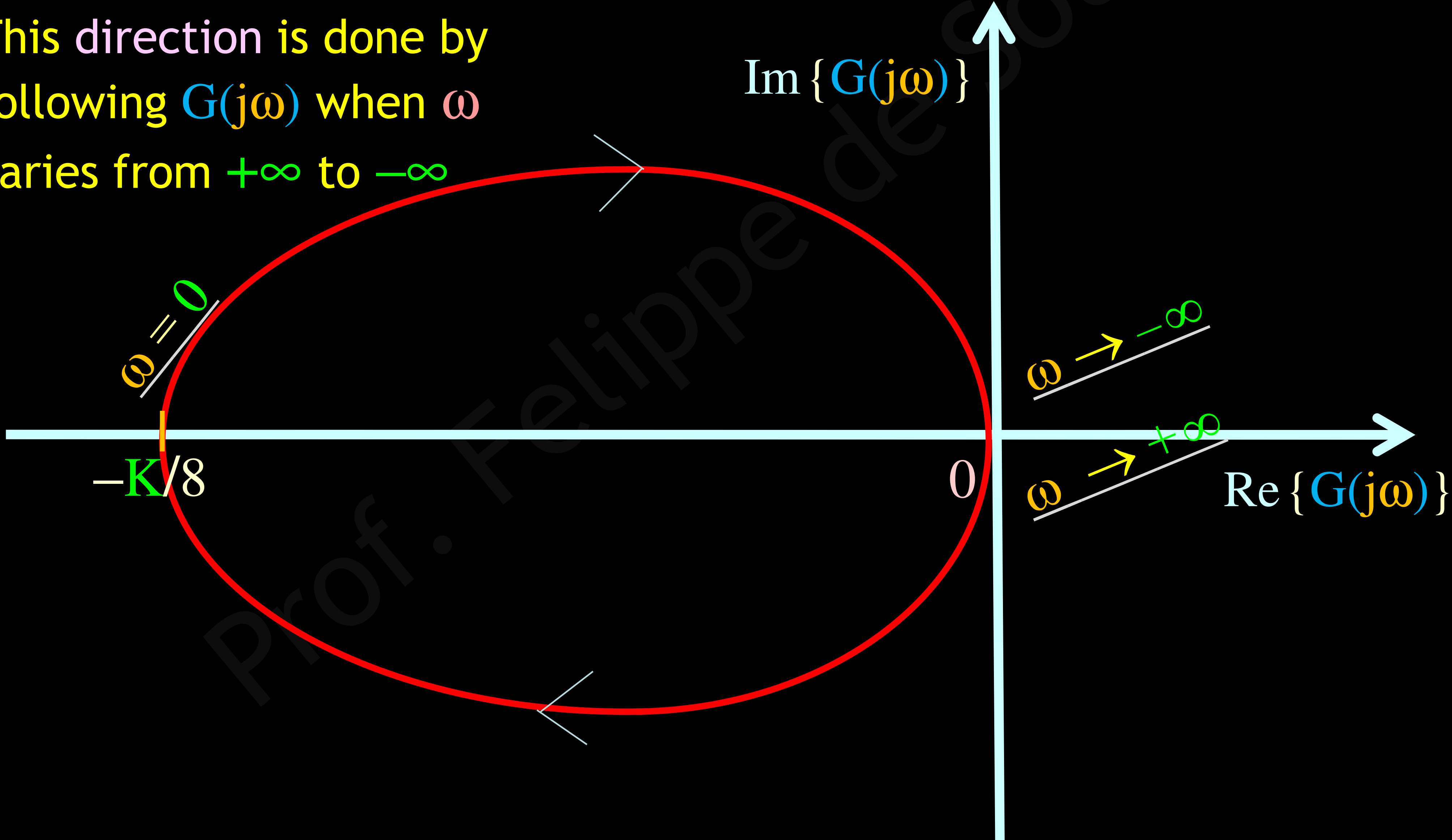
But it still missing *to give a direction* for this "*Nyquist Plot*" to become complete



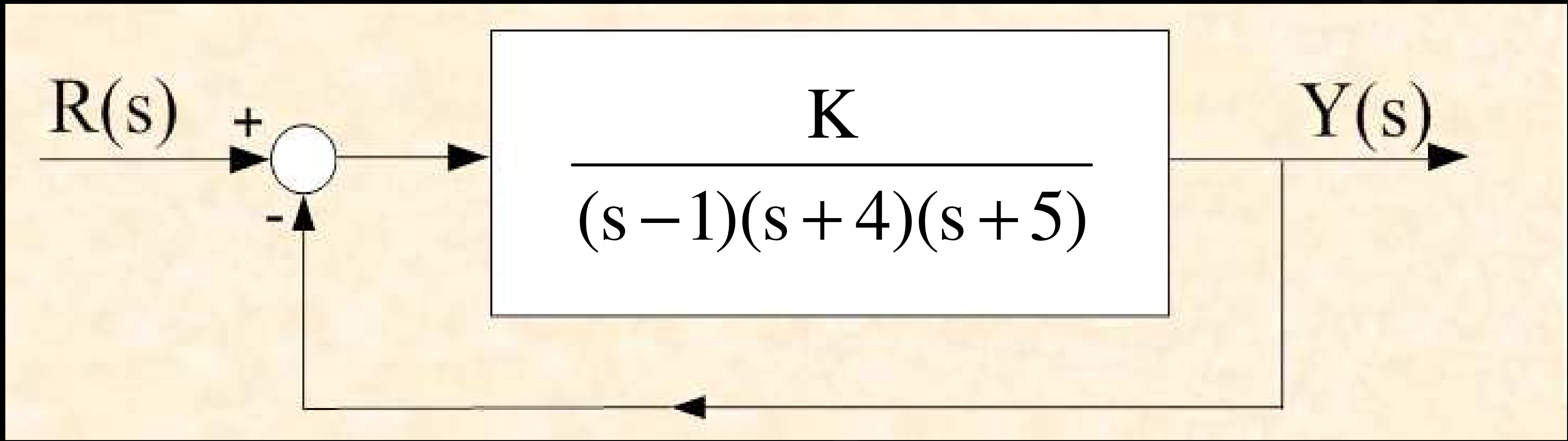
Example 1 (continued):

$$G(j\omega) = \frac{-K(\omega^2+8)}{(\omega^2+4)(\omega^2+16)} + j\frac{-2K\omega}{(\omega^2+4)(\omega^2+16)}$$

This direction is done by following $G(j\omega)$ when ω varies from $+\infty$ to $-\infty$



Example 2: Consider the closed loop *system* bellow



thus,

$$G(s) = \frac{K}{(s-1)(s+4)(s+5)}$$

and substituting $s = j\omega$, we obtain:

$$G(j\omega) = \frac{-(8\omega^2+20)\,K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)} \quad + \quad j\,\frac{(\omega^2-11)\,\omega K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)}$$

Example 2 (continued):

$$G(j\omega) = \frac{-(8\omega^2+20)\,K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)} \quad + \quad j\,\frac{(\omega^2-11)\,\omega K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)}$$

Intersection with *real axis* (*imaginary part* $= 0$)

$\omega = 0 \;\Rightarrow\; G(j\omega) = G(j0) = -K/20$

$\omega^2 = 11 \;\Rightarrow\; \omega = \pm 3.317 \;\Rightarrow\; G(j\omega) = G(\pm j3.317) = -K/108$

Intersection with *imaginary axis* (*real part* $= 0$)

$\nexists\,\omega$ real that makes null $\mathrm{Re}\{G(j\omega)\}$, the real part of $G(j\omega)$ $\Rightarrow$

➡ this Nyquist Plot **does not** intercept the *imaginary axis*

Example 2 (continued):

$$G(j\omega) = \frac{-(8\omega^2+20)\,K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)} \quad + \quad j\,\frac{(\omega^2-11)\,\omega K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)}$$

Limits at infinite ($G(j\omega)$ for $\omega = \pm\infty$)

$$G(j\infty) = 0^- + j0^+$$

(2° quadrant)

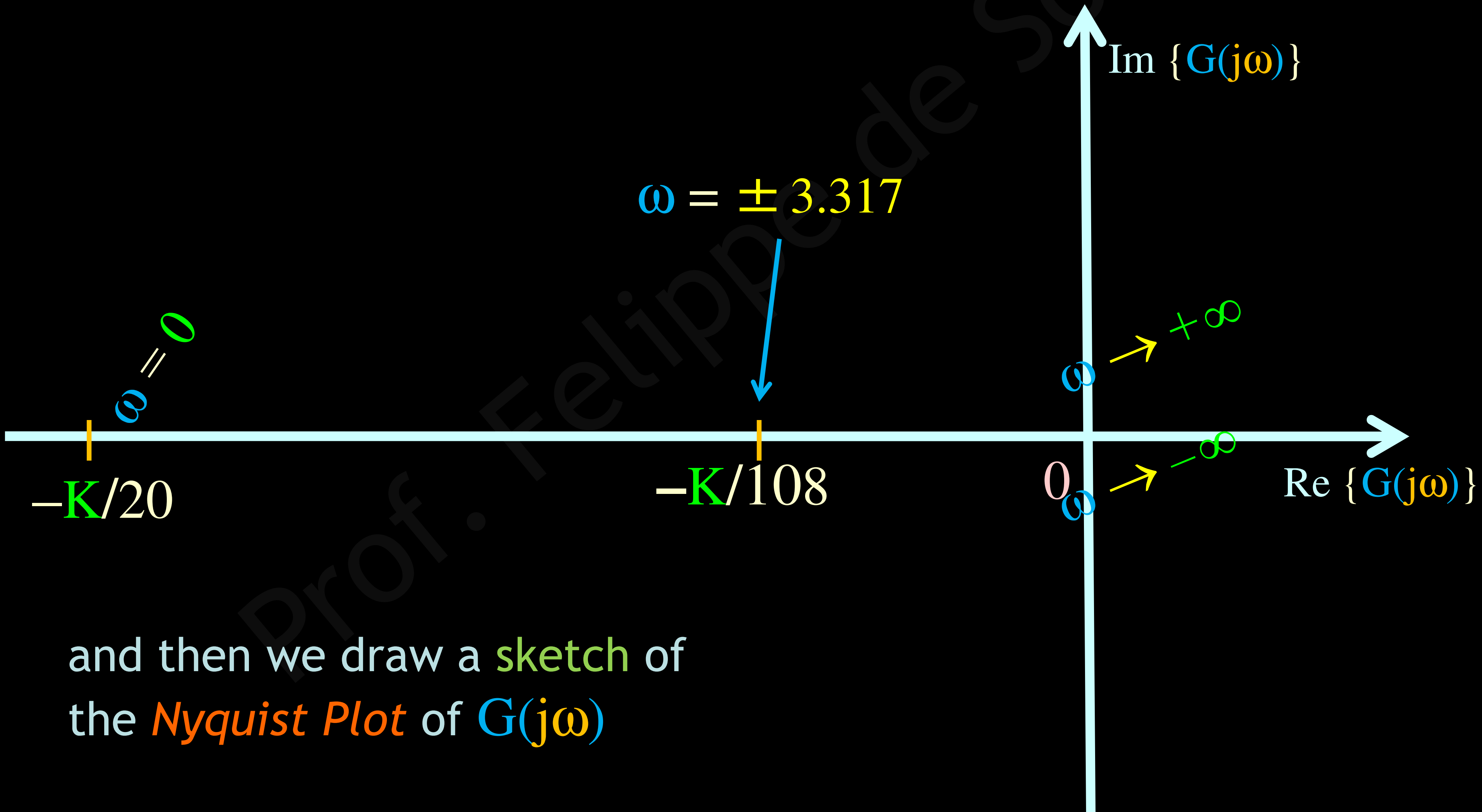
$$G(-j\infty) = 0^- + j0^-$$

(3° quadrant)

and now we write down in the complex plane
these *points of intersection* (already calculated)
and the *limits at infinite*

Example 2 (continued):

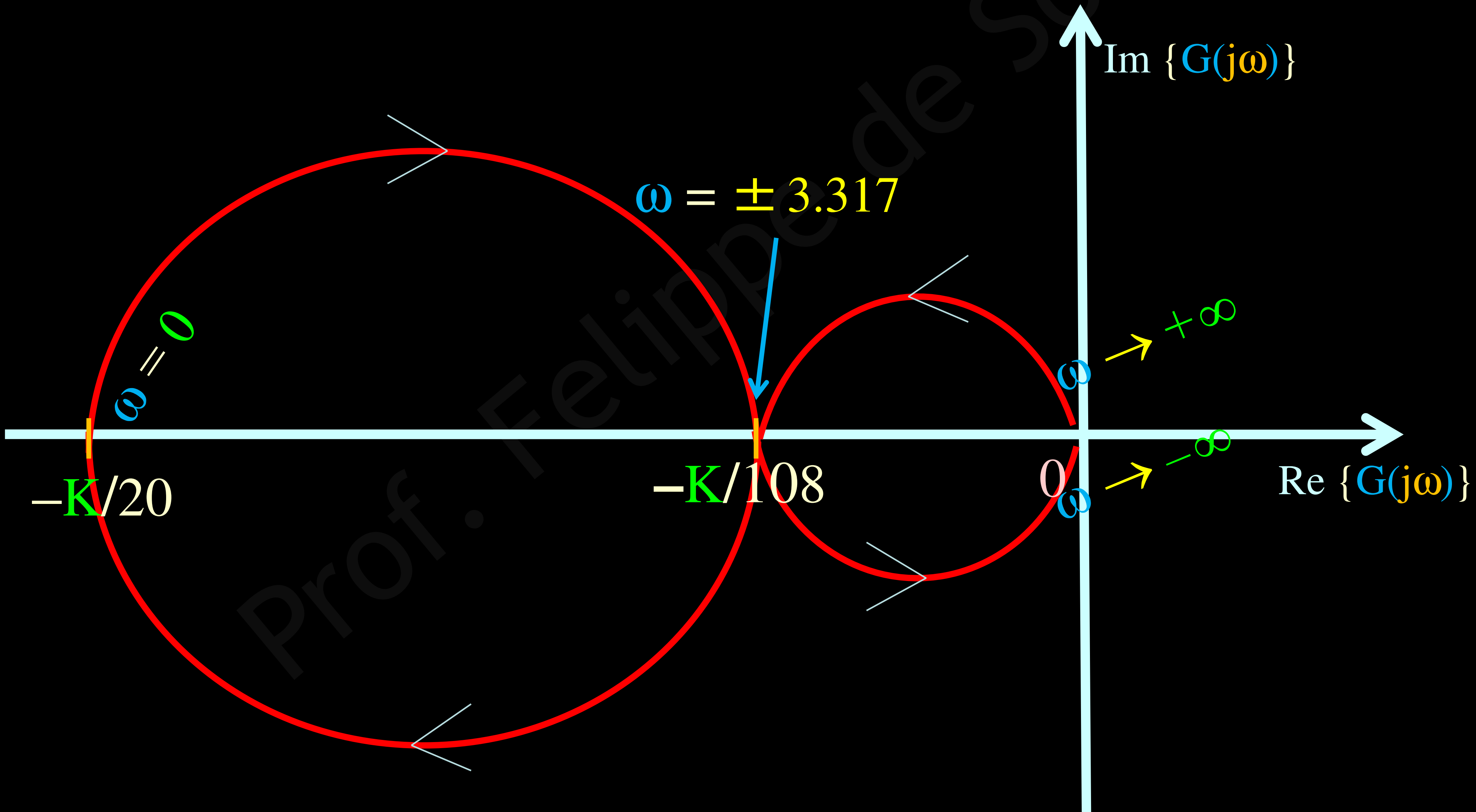
$$G(j\omega) = \frac{-(8\omega^2+20)\,K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)} \;+\; j\,\frac{(\omega^2-11)\,\omega K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)}$$



and then we draw a sketch of the *Nyquist Plot* of $G(j\omega)$

Example 2 (continued):

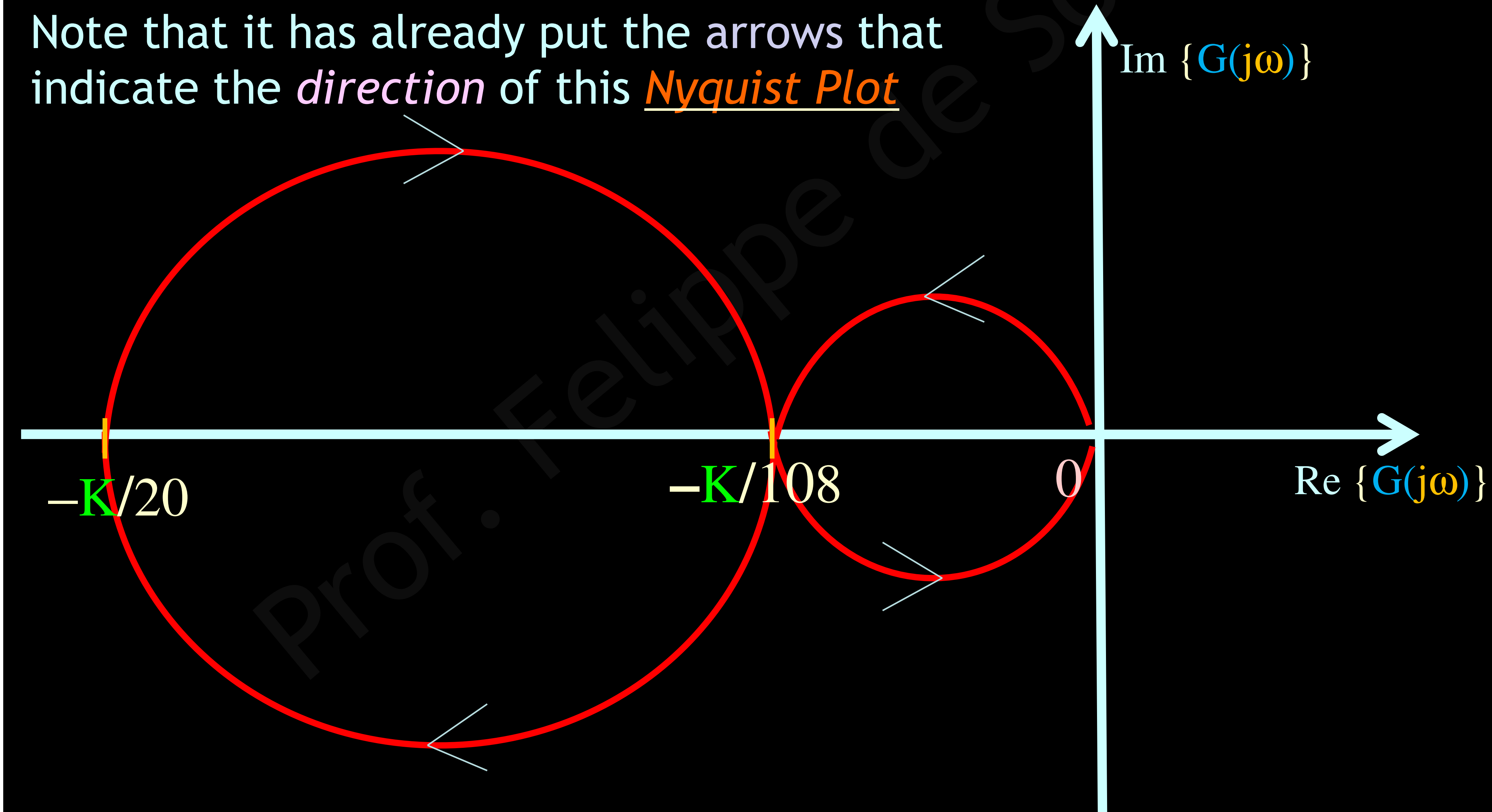
$$G(j\omega)=\frac{-(8\omega^2+20)\,K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)} \;+\; j\,\frac{(\omega^2-11)\,\omega K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)}$$



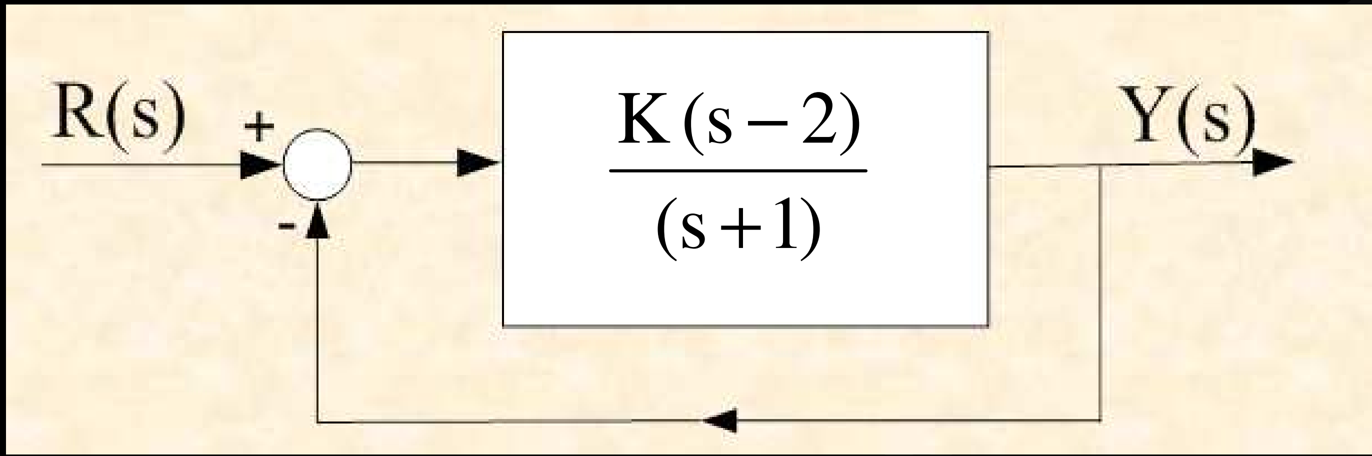
Example 2 (continued):

$$G(j\omega) = \frac{-(8\omega^2+20)\,K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)} \;+\; j\,\frac{(\omega^2-11)\,\omega K}{(\omega^2+1)\,(\omega^2+16)(\omega^2+25)}$$

Note that it has already put the arrows that indicate the *direction* of this *Nyquist Plot*



Example 3: Consider the closed loop *system* bellow:



thus,

$$G(s) = \frac{K(s-2)}{(s+1)}$$

and substituting $s = j\omega$, we obtain:

$$G(j\omega) = \frac{(\omega^2 - 2)K}{(\omega^2 + 1)} + j\frac{3\omega K}{(\omega^2 + 1)}$$

Example 3 (continued):

$$G(j\omega) = \frac{(\omega^2 - 2)K}{(\omega^2 + 1)} + j\frac{3\omega K}{(\omega^2 + 1)}$$

Intersection with *real axis* (*imaginary part* = 0)

$$\omega = 0 \;\Rightarrow\; G(j\omega) = G(j0) = -2K = -K / (½)$$

Intersection with *imaginary axis* (*real part* = 0)

$$\omega^2 = 2 \;\Rightarrow\; \begin{cases} \omega = +1.412 \;\Rightarrow\; G(j\omega) = +j\,1.412 \cdot K \\ \omega = -1.412 \;\Rightarrow\; G(j\omega) = -j\,1.412 \cdot K \end{cases}$$

Example 3 (continued):

$$G(j\omega) = \frac{(\omega^2 - 2)K}{(\omega^2 + 1)} + j\frac{3\omega K}{(\omega^2 + 1)}$$

Limits at infinite ($G(j\omega)$ for $\omega = \pm\infty$)

$$G(j\infty) = K + j0^+$$

(1º quadrant)

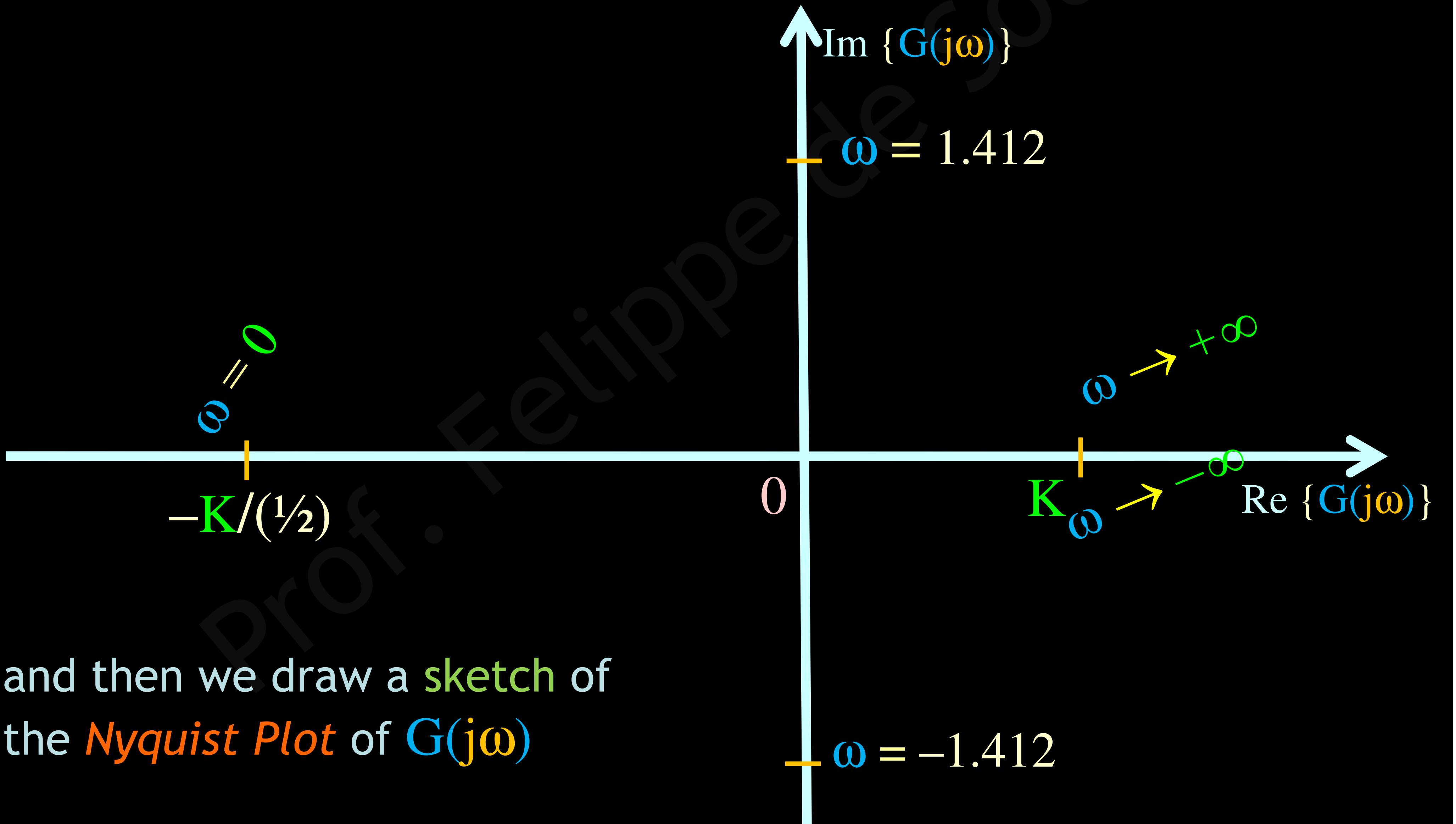
$$G(-j\infty) = K + j0^-$$

(4º quadrant)

and now we write down in the complex plane these *points of intersection* (already calculated) and the *limits at infinite*

Example 3 (continued):

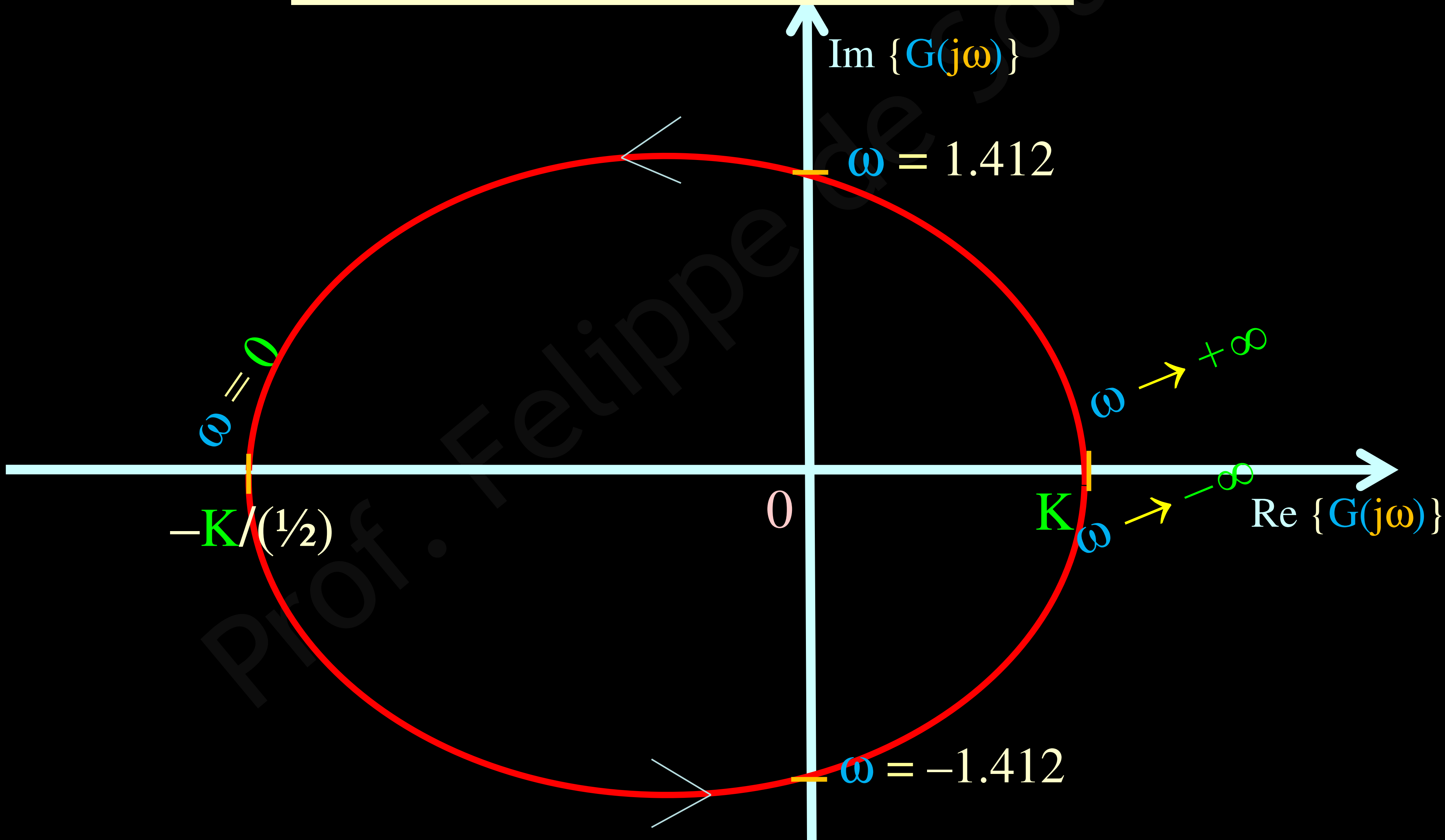
$$G(j\omega) = \frac{(\omega^2 - 2)K}{(\omega^2+1)} + j\frac{3\omega K}{(\omega^2+1)}$$



and then we draw a sketch of the *Nyquist Plot* of $G(j\omega)$

Example 3 (continued):

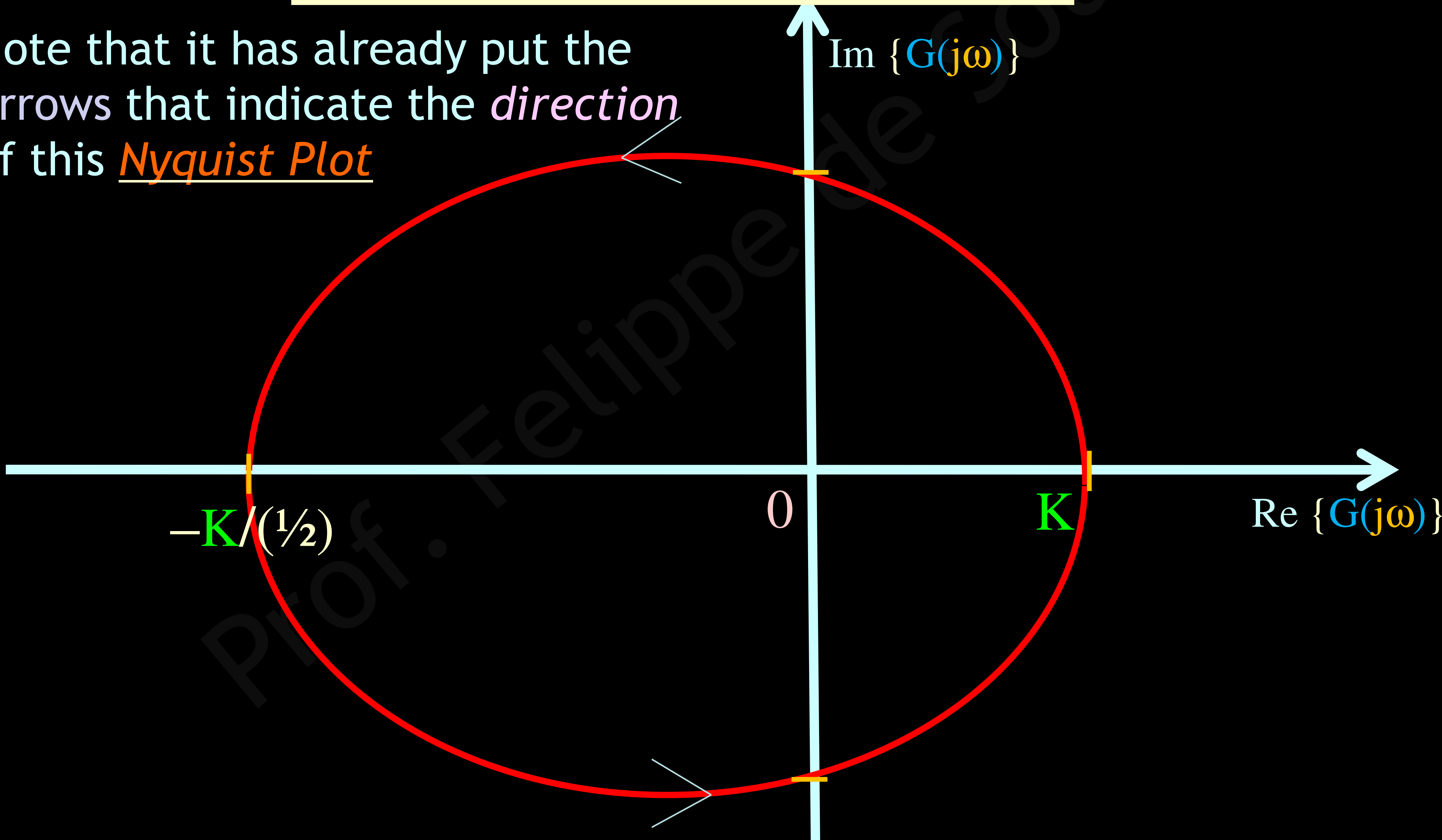
$$G(j\omega) = \frac{(\omega^2 - 2)K}{(\omega^2 + 1)} + j\frac{3\omega K}{(\omega^2 + 1)}$$



Example 3 (continued):

$$G(j\omega) = \frac{(\omega^2 - 2)K}{(\omega^2 + 1)} + j\frac{3\omega K}{(\omega^2 + 1)}$$

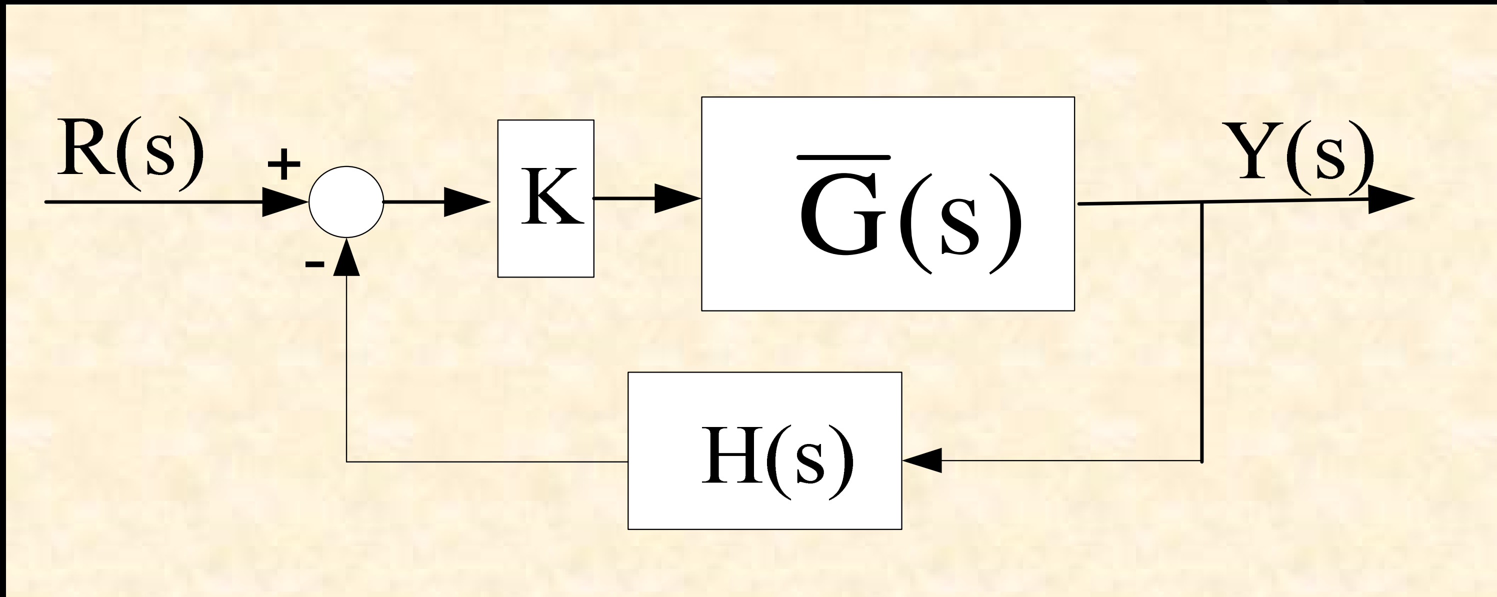
Note that it has already put the arrows that indicate the *direction* of this *Nyquist Plot*



Having the *Nyquist Plot* of $G(j\omega)$ we can apply the '*Nyquist Criterion*' to determine the stability of the M.F. *system*

Nyquist Criterion (*for stability*)

Closed loop system



Clearly, the OLTF (*open loop transfer function*) is given by:

$$G(s) = K\,\overline{G}(s)\,H(s)$$

Nyquist Criterion (*for stability*)

$$N_{-1} = P_{C.L.} - P_{O.L.}$$

- N_{-1}: Number of encirclements do Nyquist Plot around point -1
- $P_{C.L.}$: Number of open loop poles at RHP
- $P_{O.L.}$: Number of closed loop poles at RHP

N_{-1} = number of encirclements of $G(j\omega)$ around point -1, which can be '*positive*' (if direction is *counter clockwise*), or '*negative*' (if direction is *clockwise*)

Nyquist Criterion (*for stability*)

that is,

$$P_{C.L.} = N_{-1} + P_{O.L.}$$

$P_{C.L.}$ = Number of *closed loop* poles at RHP

N_{-1} = Number of encirclements of the Nyquist Plot around point -1

$P_{O.L.}$ = Number of *open loop* poles at RHP

And, of course, the *closed loop system* will be *stable* if

$$P_{M.F.} = 0$$

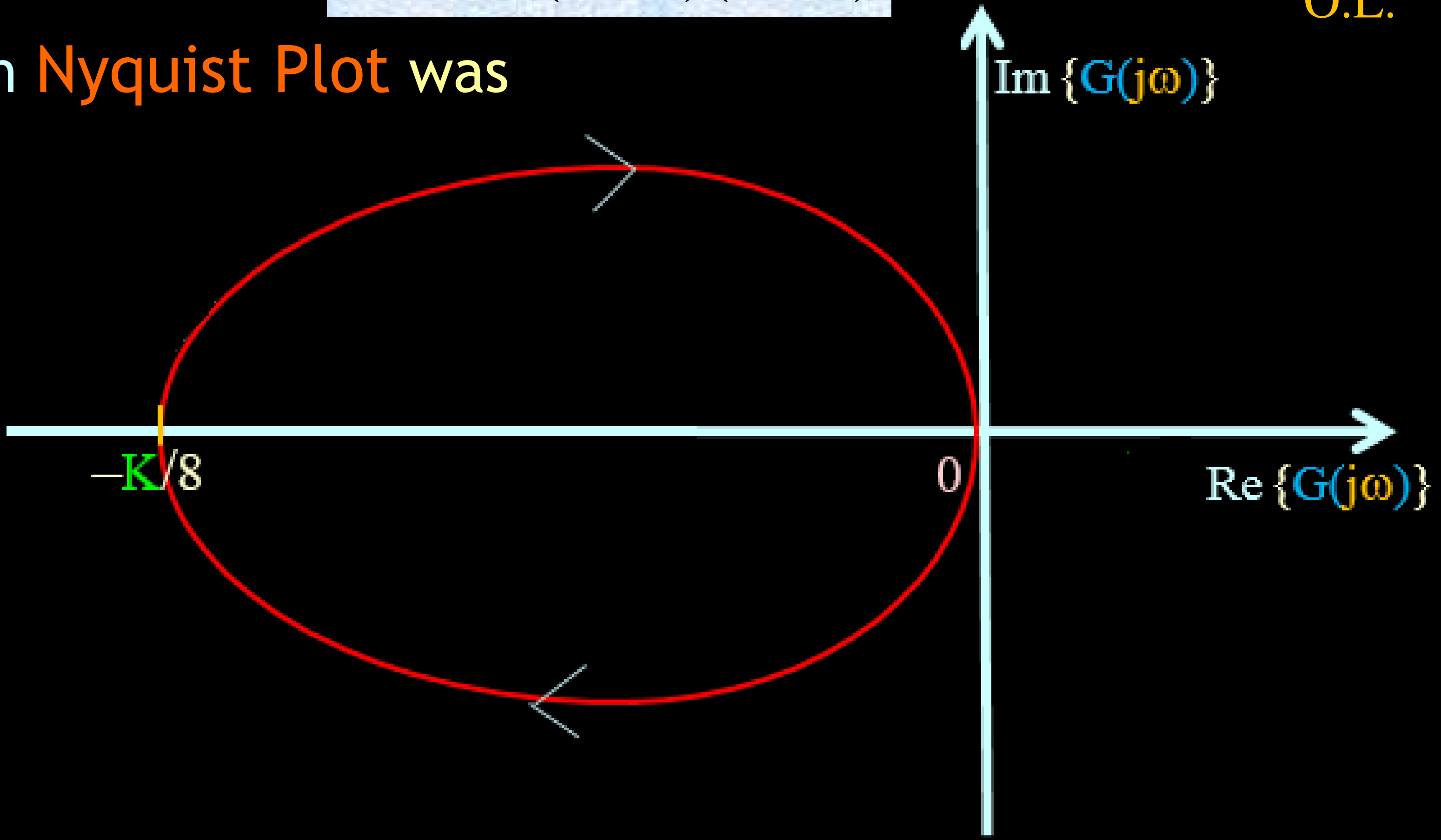
Application of Nyquist Criterion to system from Example 1

Example 1 (continued): Let us return to this problem

$$G(s) = \frac{K}{(s-2)(s+4)}$$

then, $P_{O.L.} = 1$

which Nyquist Plot was

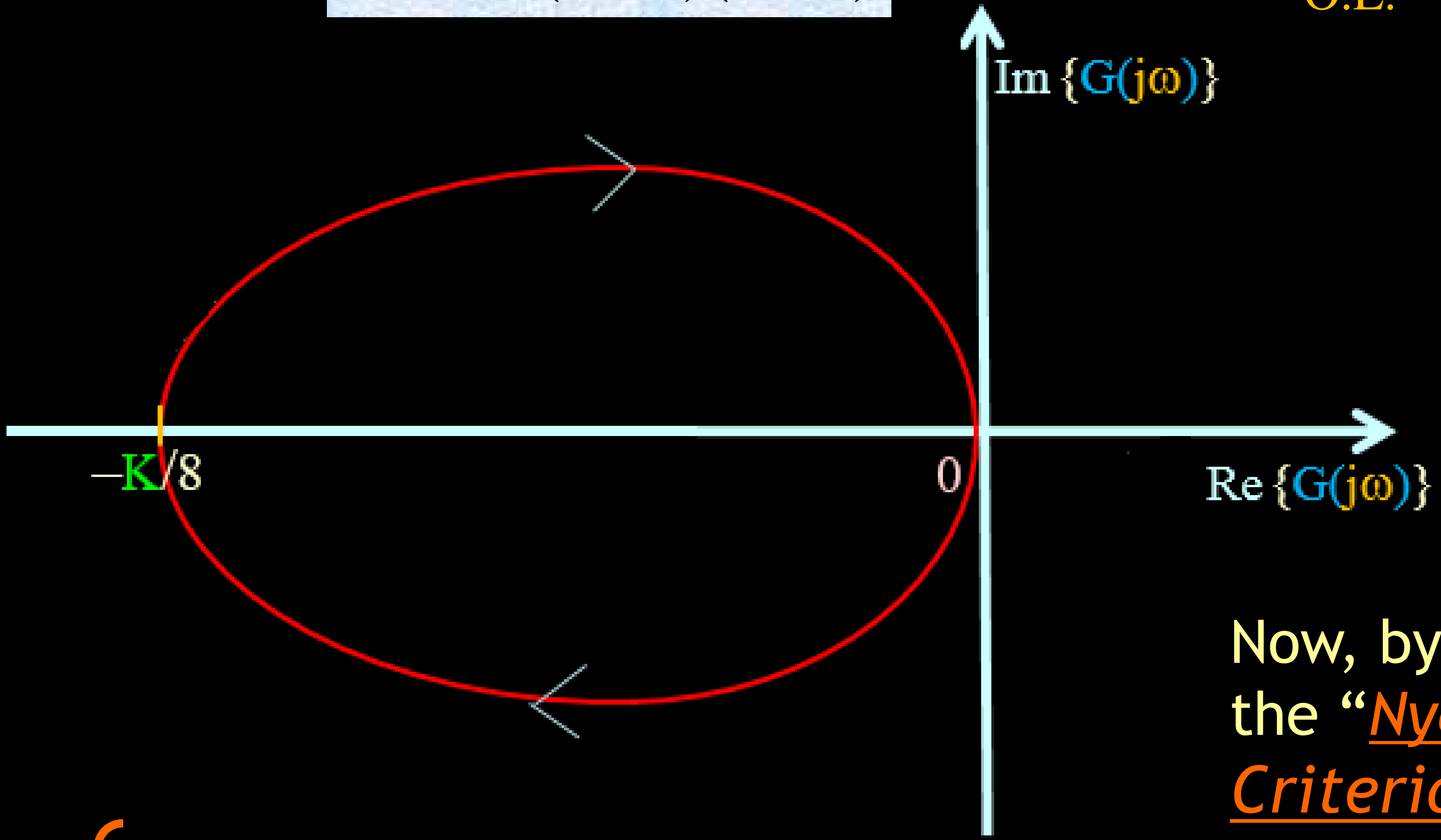


By doing now an analysis to the number of encirclements of the Nyquist Plot around point -1, we get

Example 1 (continued):

$$G(s) = \frac{K}{(s-2)(s+4)}$$

$P_{O.L.} = 1$



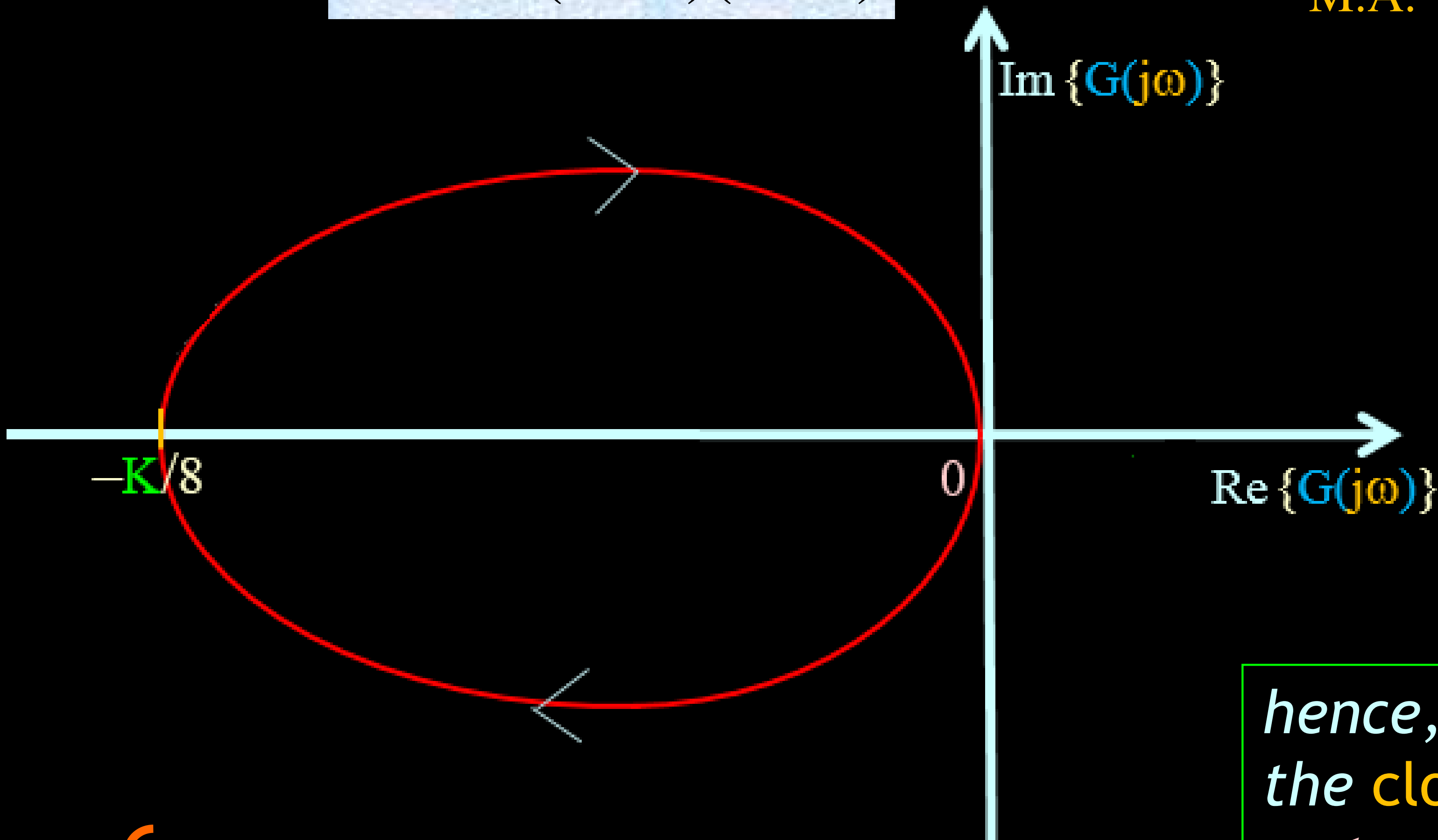
Now, by applying the "*Nyquist Criterion*" to determine the C.L. *stability*, we get

$$N_{-1} = \begin{cases} 0 & \text{se} \quad K < 8 \\ -1 & \text{se} \quad K > 8 \end{cases}$$

Example 1 (continued):

$$G(s) = \frac{K}{(s-2)(s+4)}$$

$$P_{M.A.} = 1$$



$$P_{M.F.} = \begin{cases} 0 + 1 = 1 & \text{se} \quad K < 8 \\ -1 + 1 = 0 & \text{se} \quad K > 8 \end{cases}$$

hence,
the closed loop
system is *stable*
for

$K > 8$

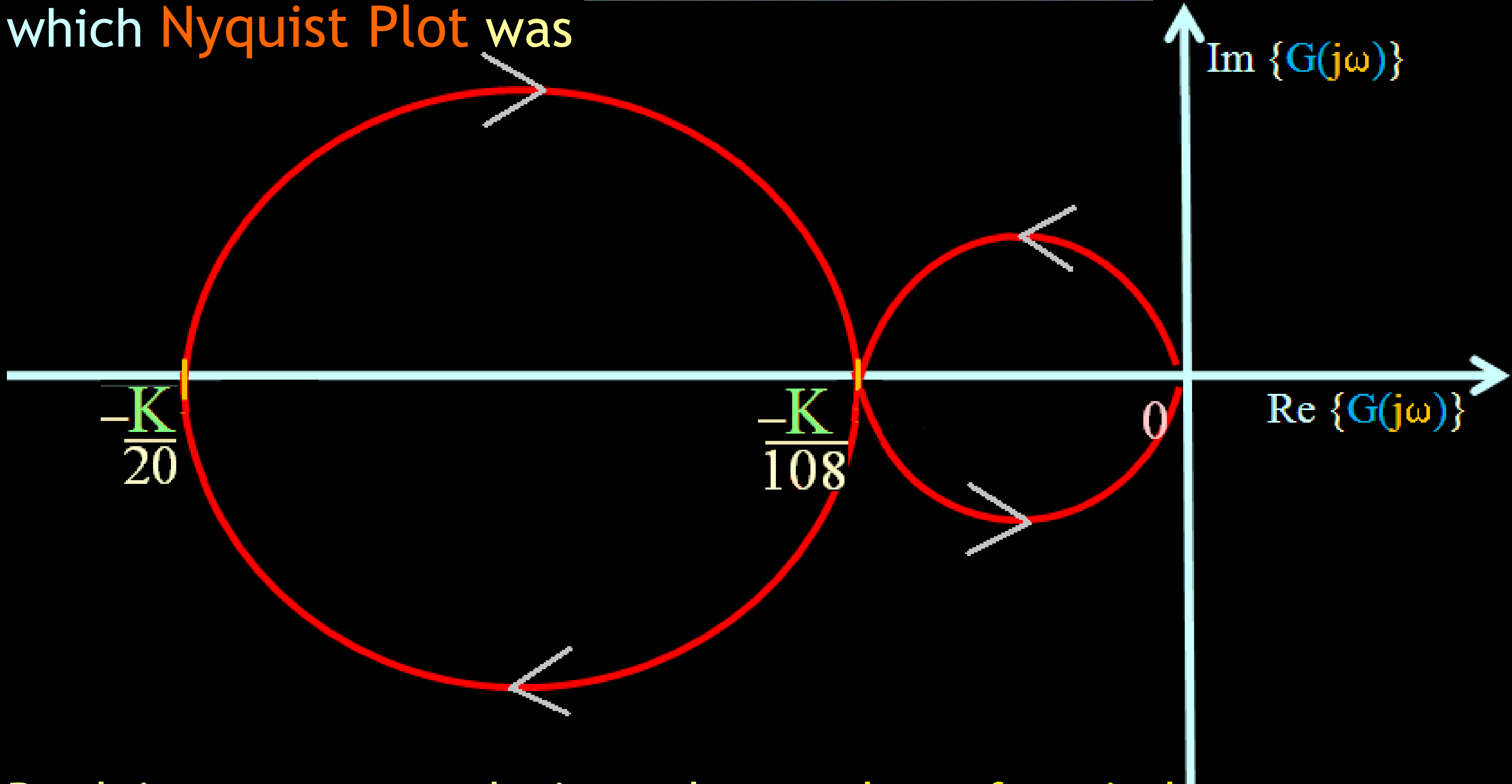
Application of Nyquist Criterion
to system from Example 2

Example 2 (continued): Let us now return to this problem

$$G(s) = \frac{K}{(s-1)(s+4)(s+5)}$$

then, $P_{M.A.} = 1$

which Nyquist Plot was

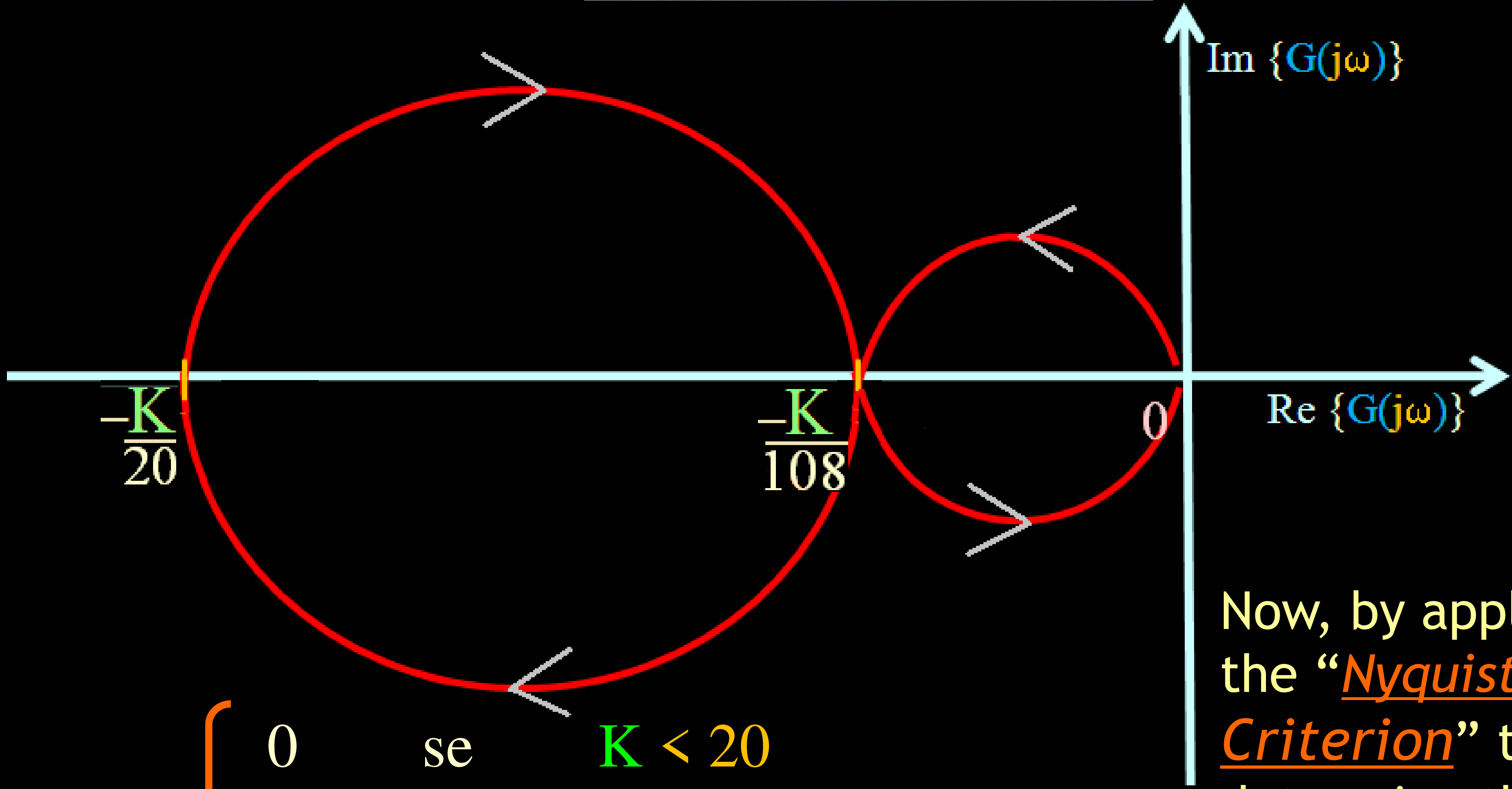


By doing now an analysis to the number of encirclements of the Nyquist Plot around point -1, we get

Example 2 (continued):

$$G(s) = \frac{K}{(s-1)(s+4)(s+5)}$$

$$P_{M.A.} = 1$$



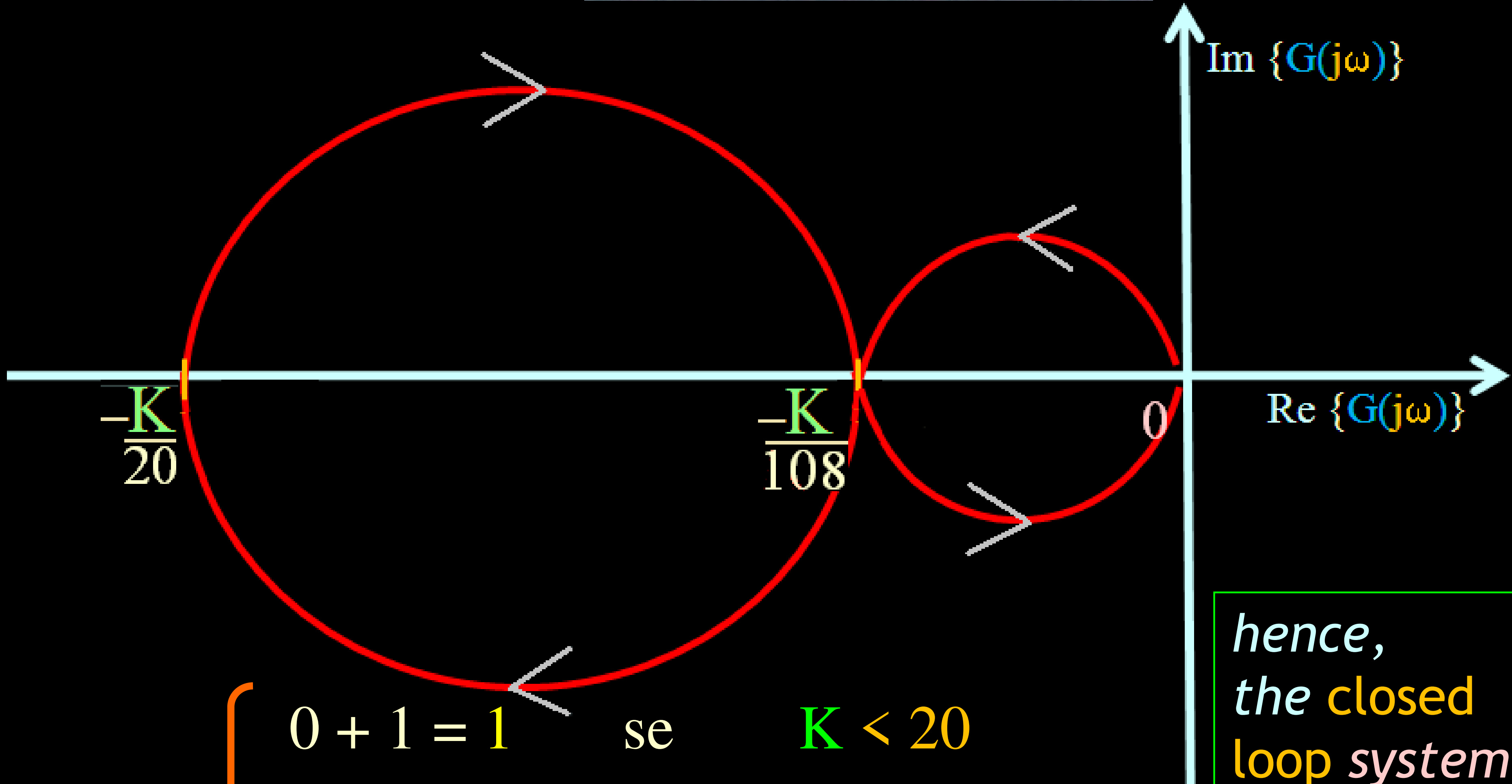
Now, by applying the "*Nyquist Criterion*" to determine the C.L. *stability*, we get

$$N_{-1} = \begin{cases} 0 & \text{se} \quad K < 20 \\ -1 & \text{se} \quad 20 < K < 108 \\ 1 & \text{se} \quad K > 108 \end{cases}$$

Example 2 (continued):

$$G(s) = \frac{K}{(s-1)(s+4)(s+5)}$$

$P_{M.A.} = 1$



$$P_{M.F.} = \begin{cases} 0 + 1 = 1 & \text{se} \quad K < 20 \\ -1 + 1 = 0 & \text{se} \quad 20 < K < 108 \\ 1 + 1 = 2 & \text{se} \quad K > 108 \end{cases}$$

hence, *the* closed loop *system* is *stable* for $20 < K < 108$

Application of Nyquist Criterion
to system from Example 3

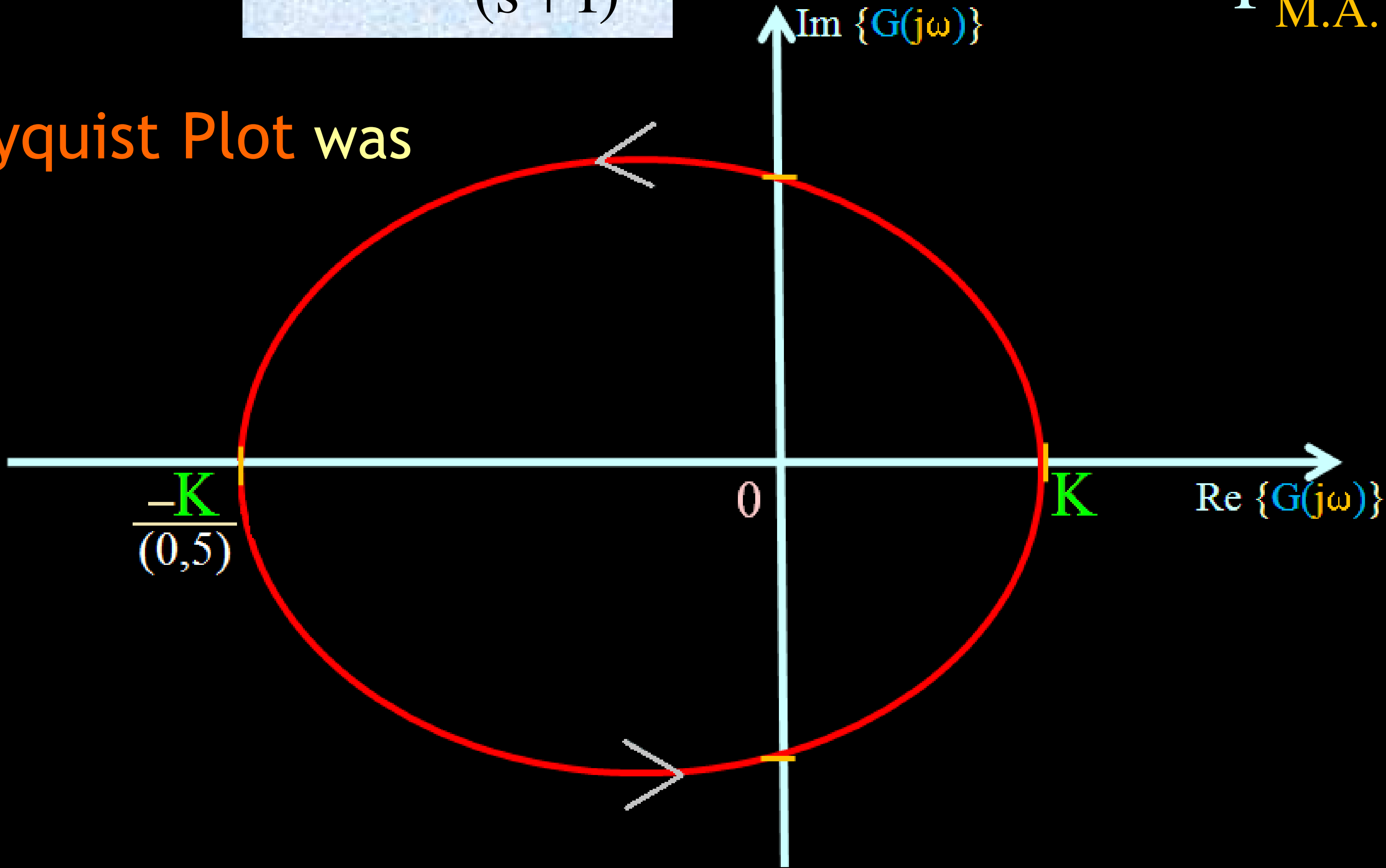
Example 3 (continued): Finally, let us now return to this problem

$$G(s) = \frac{K(s-2)}{(s+1)}$$

then,

$$P_{M.A.} = 0$$

which Nyquist Plot was

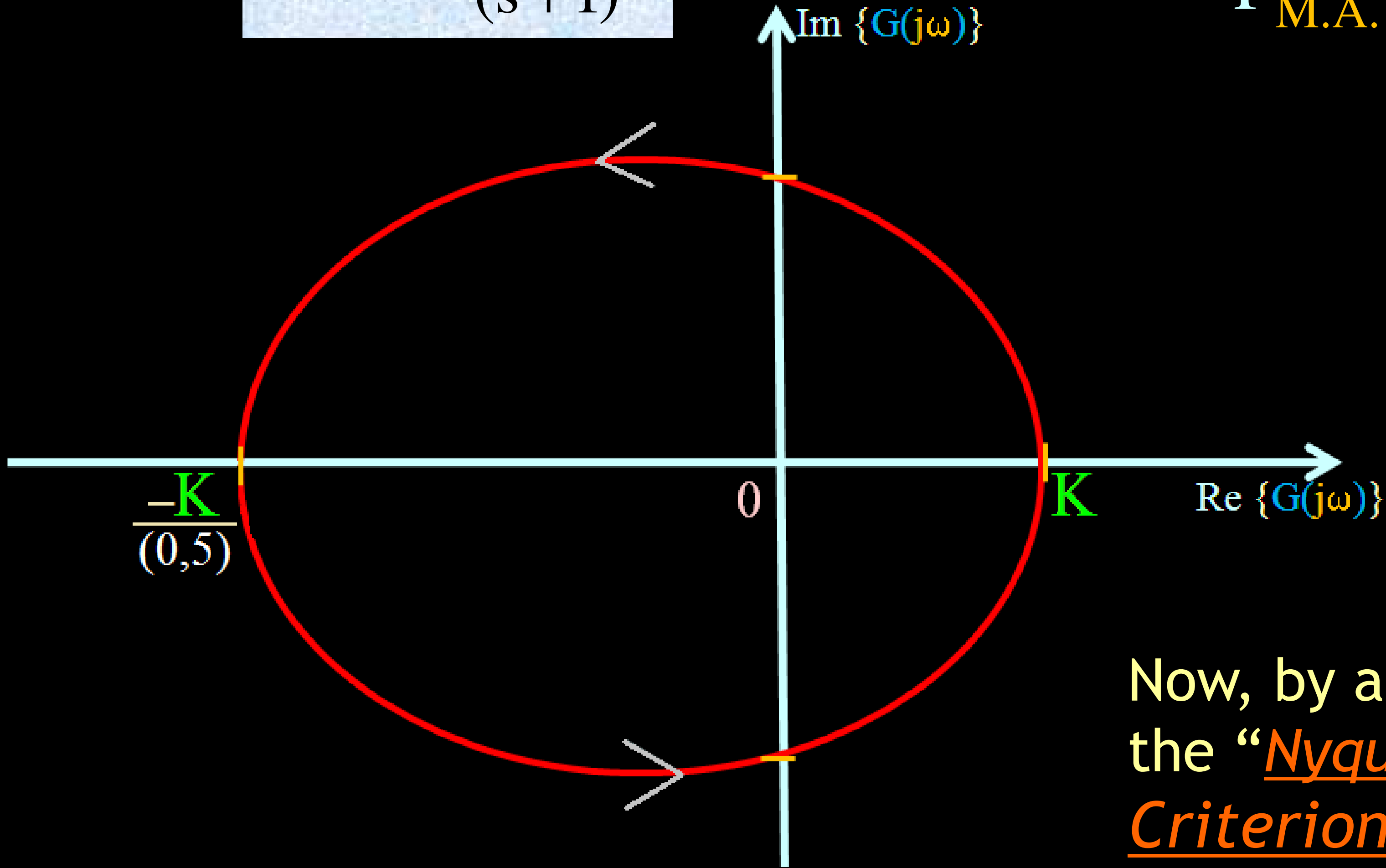


By doing now an analysis to the number of encirclements of the Nyquist Plot around point -1, we get

Example 3 (continued):

$$G(s) = \frac{K\,(s-2)}{(s+1)}$$

$$P_{M.A.} = 0$$



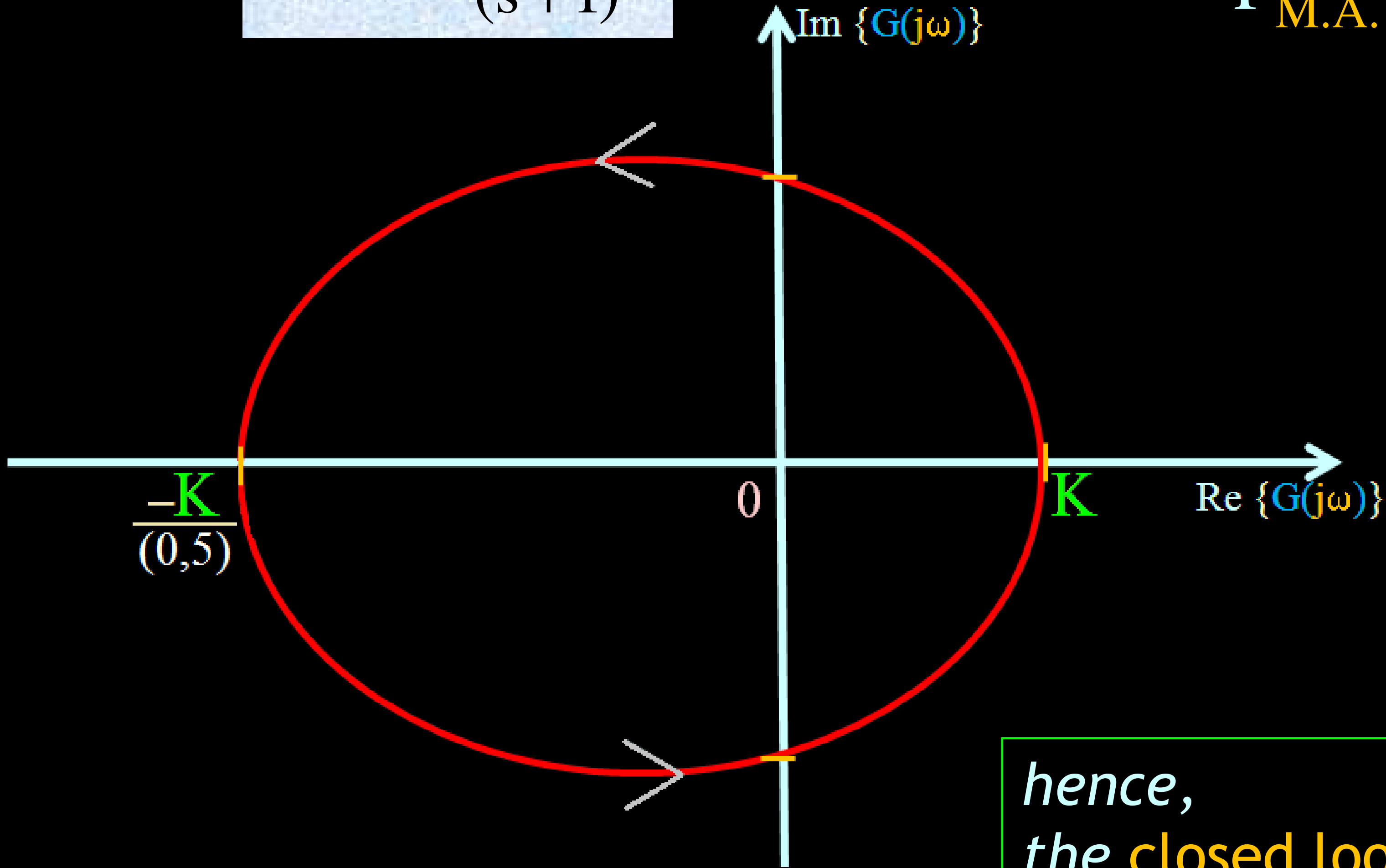
Now, by applying the "*Nyquist Criterion*" to determine the C.L. *stability*, we get

$$N_{-1} = \begin{cases} 0 & \text{se} \quad K < 0{,}5 \\ 1 & \text{se} \quad K > 0{,}5 \end{cases}$$

Example 3 (continued):

$$G(s) = \frac{K(s-2)}{(s+1)}$$

$P_{M.A.} = 0$



$$P_{M.F.} = \begin{cases} 0 + 0 = 0 & \text{se} \quad K < 0{,}5 \\ 1 + 0 = 1 & \text{se} \quad K > 0{,}5 \end{cases}$$

hence,
the closed loop
system is *stable* for
$K < 0{,}5$

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Obrigado!

Thank you!

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