Control Systems

12

“Root Locus”

part II

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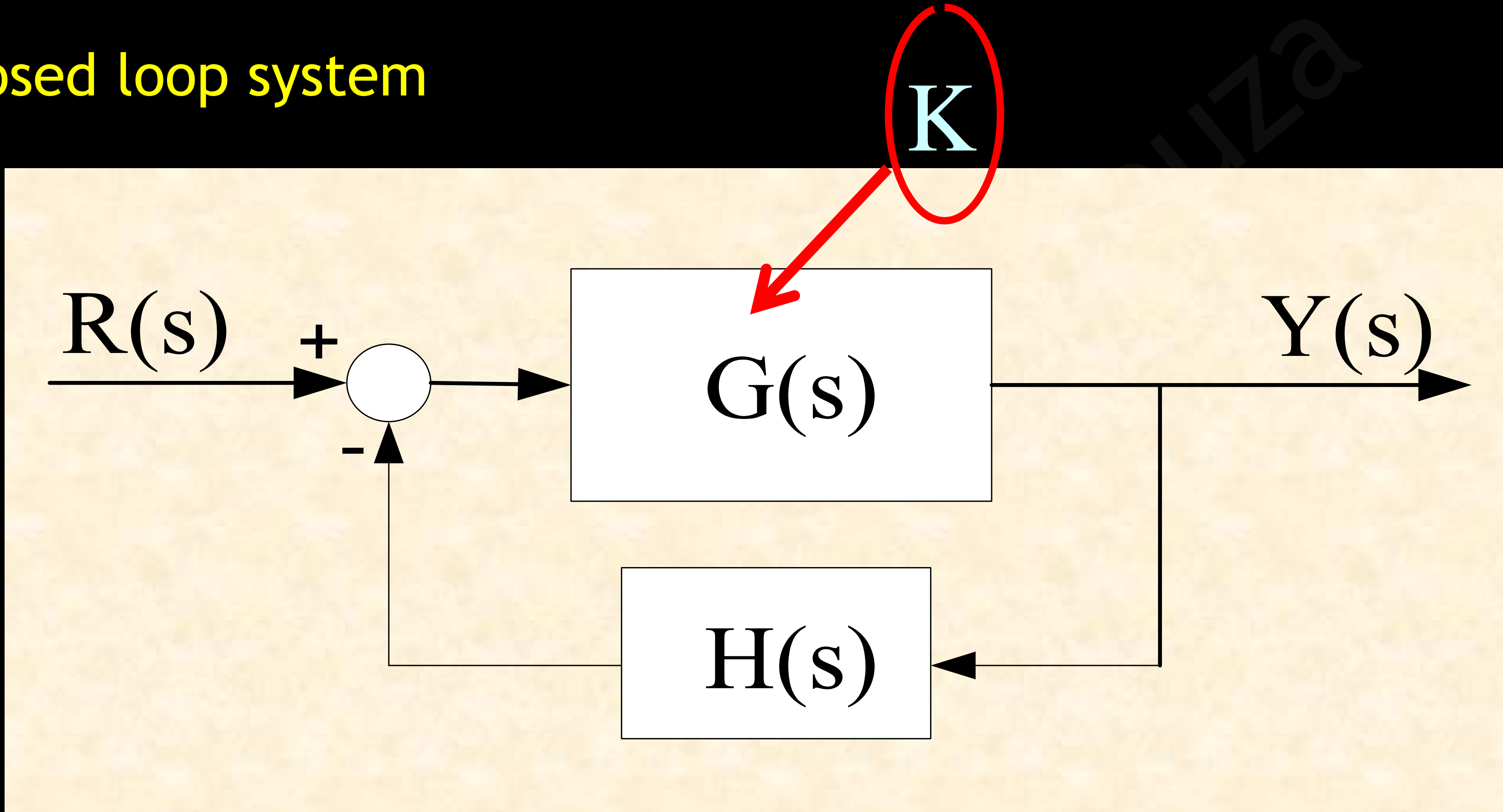
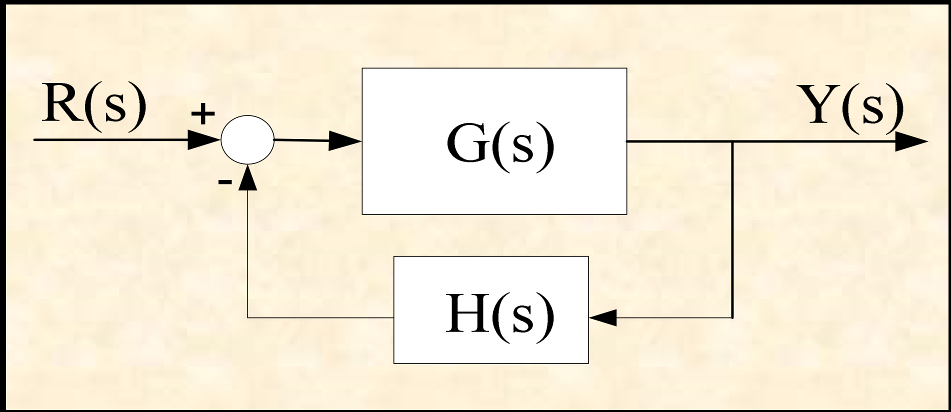
Revising (*from part I*):

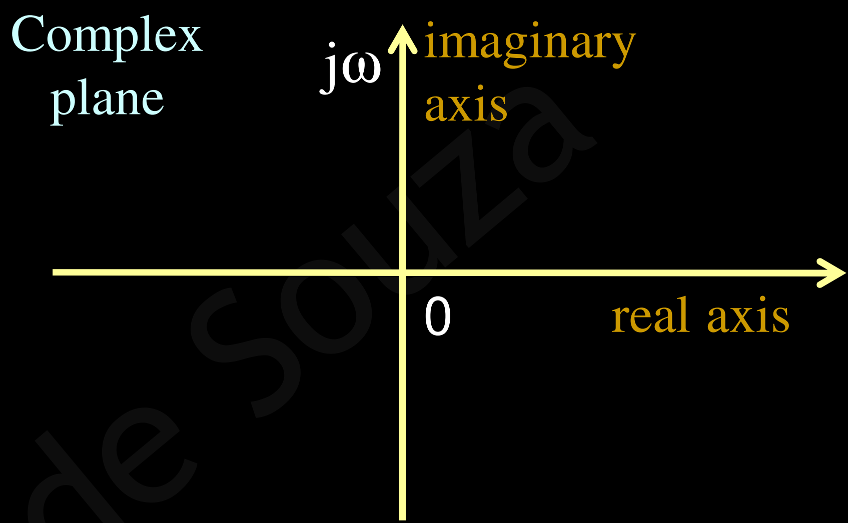
Closed loop system



The "*Root Locus*" the locus of the *poles* of the *closed loop* system, when we vary the value of K

Root Locus part II





Thus, the "*Root Locus*" is drawn in the *complex plane*

It is easy to observe that the "*Root Locus*" is SIMETRIC with respect to the *real axis*

That is, the *upper part* is a reflex of the *lower part*

It is easy to show that these *roots* of the *characteristic equation* of the CLTF are the same *roots* of

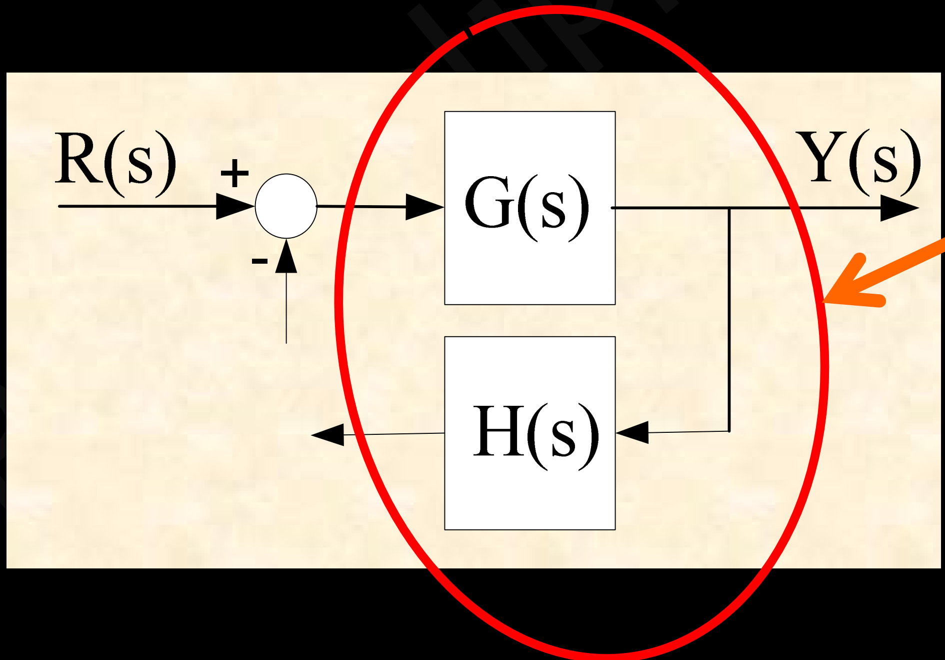
$$1 + G(s) \cdot H(s) = 0$$

Note that the "*Root Locus*" depend only on the produt $G(s)\cdot H(s)$ and not in $G(s)$ or $H(s)$ separately

The expression

$$G(s)\cdot H(s)$$

Is called *transfer function* of the *system* in *open loop* (OLTF).



$G(s)\cdot H(s)$
(OLTF)

Rules for the constructing the "*Root Locus*"

We shall call

n = the number of *open loop poles*

m = the number of *open loop zeros*

Rule #1 – The number of branches

The number of branches n of a "*Root Locus*" is the number of open loop *poles*, that is, the number of *poles* of $G(s)\cdot H(s)$.

$$\boxed{n \;=\; \text{nº } \textit{branches} \;=\; \text{nº } \textit{poles} \text{ de } G(s)\cdot H(s)}$$

Rule #2 – Intervals with and without "*Root Locus*" in the *real axis*

A point s in the *real axis* belongs to the "*Root Locus*" if there is an odd number of *open loop poles* and *zeros* which are real and located to the right of s

that is, if there is an odd number of *poles* and *zeros* of $G(s)\cdot H(s)$ which are real and located to the right of s

Rule #3 - Beginning and end points of the branches of the "*Root Locus*"

The n branches of the "*Root Locus*" begins in the n de open loop *poles*

that is, they start at the n *poles* of $G(s)\cdot H(s)$

m of the n branches of the "*Root Locus*" end in the m open loop *zeros*

that is, they finish in the m *zeros* of $G(s)\cdot H(s)$

and the remainders:

$(n - m)$ branches of the "*Root Locus*" finish in the *infinite* (∞)

Rule #4 - Asymptotes in the infinite

For the $(n - m)$ branches of the "*Root Locus*" that do not end at the m *open loop zeros*, that is, the m *finite zeros* of $G(s) \cdot H(s)$, one can determine a direction that they go to *infinite* in the *complex plane*.

γ = angle of the asymptote with the *real axis*

$$\boxed{\gamma = \frac{180^\circ \cdot (2i + 1)}{(n - m)}} \qquad \boxed{i = 0, 1, 2, \cdots}$$

Rule #5 – Point of intersection of the asymptotes with the *real axis*

The $(n-m)$ asymptotes in the infinite are well determined by its directions (*angles* γ) and by the *point* from where they leave the *real axis*, σ_o given by the expression:

$$\sigma_o = \frac{\left(\sum_{i=1}^{n} \mathrm{Re}(p_i) - \sum_{j=1}^{m} \mathrm{Re}(z_j) \right)}{(n-m)}$$

Rule #6 – Points of the *real axis* where there are *branches* crossing

First the equation is found

$$1 + G(s)\cdot H(s) = 0,$$

thus the expression for K is calculated as function of s:

$$K(s)$$

then the derivative of K with respect to s, dK/ds is found,

Now, using the equation below in s

$$\boxed{\frac{dK}{ds} = 0}$$

we obtain the points s of the *real axis* where the *branches* meet

Rule #7 – Encounter of more than two branches

When applying the *previous rule*, if

$$\left.\frac{d^2K}{ds^2}\right|_{s=s'} = 0$$

this means that there are encounter of *more than two branches* and we have to keep on the differentiation on $K(s)$, to higher order *derivatives*

$$\frac{d^kK}{ds^k} \qquad k = 3, 4, 5, \cdots$$

until we get

$$\left.\frac{d^\eta K}{ds^\eta}\right|_{s=s'} \neq 0 \qquad \text{for some } \eta$$

Rule #7 – Encounter of more than two branches (*continued*)

If $\boxed{\left.\dfrac{d^{\eta}K}{ds^{\eta}}\right|_{s=s'} \neq 0}$ and $\boxed{\left.\dfrac{d^{k}K}{ds^{k}}\right|_{s=s'} = 0}$ for $\forall\ k < \eta$

this means that there is a meeting of η *branches* in s'

that is, η *branches* arrive and η *branches* leave s'

In this part II we will see the *last rule* (Rule #8) and some examples of *Root Locus*

Rule #8

Crossing points of the "*Root Locus*" with the *imaginary axis*

Rule #8 – Crossing points of the "*Root Locus*" with the *imaginary axis*

The use of Routh-Hurwitz table with the *closed loop characteristic equation* of the *system* which is obtained from

$$1 + G(s) \cdot H(s) = 0$$

Note that the *closed loop characteristic equation* obtained will be in function of K, and with it one can form the Routh-Hurwitz table to apply the Routh-Hurwitz stability criterion

To apply the Routh-Hurwitz stability criterion we have to find the values of K that make the *pivot column* elements of the Routh-Hurwitz table to vanish

Example 9: Routh-Hurwitz table for a *characteristic equation* as function of K obtained through $1 + G(s)\cdot H(s) = 0$

$$p(s) = s^4 + s^3 + Ks^2 + 3s + (K-10)$$

s^4	1	K	$(K-10)$	
s^3	1	3	0	← etc…
s^2	$(K-3)$	$(K-10)$	← 2nd line before the last	
s^1	$(2K+1)/(K-3)$	← line before the last		
s^0	$(K-10)$	← last line		

↑ *pivot column*

Clearly, the only values of $K > 0$ that makes an element of the *pivot column* to vanish are

$K = 3$ (in the 2nd line before the last) and

$K = 10$ (in the last line)

Rule #8 – Crossing points of the "*Root Locus*" with the *imaginary axis* (*continued*)

If $\nexists$ $K > 0$ that makes elements *pivot column* to vanish

$\Rightarrow$ "*Root Locus*" does not intercept imaginary axis.

If $\exists$ $K > 0$ that makes to vanish the LAST element *pivot column*

$\Rightarrow$ "*Root Locus*" intercepts imaginary axis in *one point*, *at origin* ($s = 0$).

If $\exists$ $K > 0$ that makes to vanish the element before the LAST *pivot column*

$\Rightarrow$ "*Root Locus*" can intercept imaginary axis in *two points* ($s = \pm j\omega'$).

If $\exists$ $K > 0$ that makes to vanish the 2^{nd} element before LAST *pivot column*

$\Rightarrow$ "*Root Locus*" can intercept imaginary axis in *three points* ($s = 0$ e $s = \pm j\omega'$).

and so forth …

Rule #8 – Crossing points of the "*Root Locus*" with the *imaginary axis* (*continued*)

In order to know the exact points where the "*Root Locus*" intercepts the *imaginary axis* we need to rewrite the *characteristic equation* $p(s)$ substituting K for each of the values of K that makes the elements of *pivot column* to vanish

After that, we calculate the roots of $p(s)$

In it we will find the eventual *crossing points* of the "*Root Locus*" with the *imaginary axis*.

If however, instead of $p(s)$ we calculate the *polynomial* of the *line immediately above* where K makes vanish the *pivot column* (in *Routh-Hurwitz Table*), will give us the *crossing points* of the "*Root Locus*" with the *imaginary axis*.

Example 10

Application of Rule #8 –
Crossing points of the "*Root Locus*" with the *imaginary axis*

Returning to Example 1 (part I), the *characteristic equation* of the CLTF is given by:

$$s^2 + (2K–4)\,s + K = 0$$

Setting up the *Routh-Hurwitz table* (as function of K):

$$\begin{array}{c|ll}
s^2 & 1 & K \\
s^1 & (2K-4) & \Longrightarrow (2K-4)=0 \Longrightarrow \boxed{K=2} \\
s^0 & K & \Longrightarrow \boxed{K=0}
\end{array}$$

Analysing where there are $K \geq 0$ that makes the elements of *pivot column* to vanish, we have that "*Root Locus*" intercepts the *imaginary axis* in *3 points*:

at the origin for $\boxed{K = 0}$ and also in $\pm j\omega'$ when $\boxed{K = 2}$

Example 10 (*continued*) Application of Rule #8

Now, in order to find the 3 exact values where the "*Root Locus*" intercepts the *imaginary axis*:

For $K = 0$ (*that makes last line to vanish*), we take polynomial of the line immediately above (that is, the *line before the last*) and calculate the roots (in this case the root):

$$p(s) = (2K - 4)\,s = -4s = 0$$ ⇒ root: $s = 0$

(*as previously seen*)

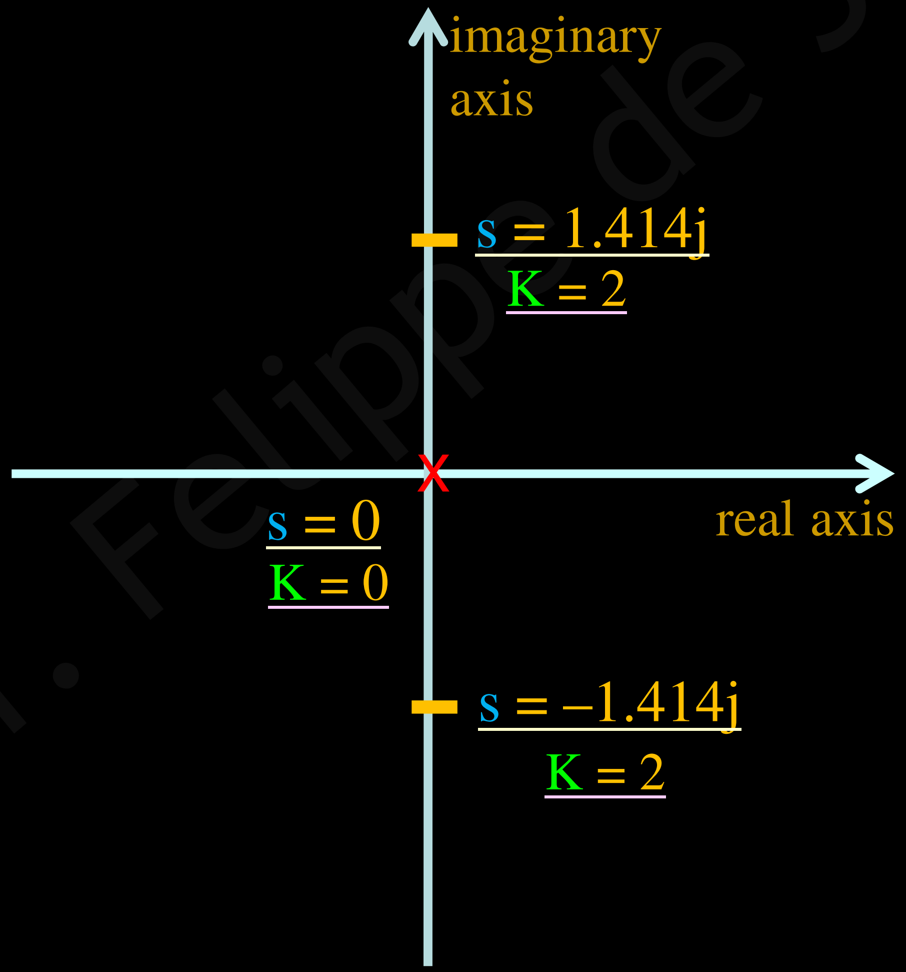
For $K = 2$ (*that makes the line before the last to vanish*), we take polynomial of the line immediately above (that is, the *2nd line before the last*) and calculate the roots:

$$p(s) = s^2 + K = s^2 + 2 = 0$$

⇒ roots: $s = \pm\, 1.414j$ (thus, $\omega' = 1.414$)

Example 10 (*continued*) Application of Rule #8

Concluding, this "*Root Locus*" intercepts the *imaginary axis* in the 3 points shown below



Example 11 Application of Rule #8 –

Crossing points of the "*Root Locus*" with the *imaginary axis*

Consider the *system* which CLTF is given by

$$G(s)H(s) = \frac{K \cdot (s-2)^2}{(s^2+2s+2)\cdot(s+1)}$$

by doing:

$$1 + G(s)H(s) = 0$$

we obtain the *characteristic equation* of the CLTF:

$$s^3 + (K+3)\,s^2 + (4-4K)\,s + (4K+2) = 0$$

Example 11 (*continued*) Application of Rule #8

By setting up the Routh-Hurwitz table (as function of K):

s^3	1	$(4-4K)$
s^2	$(K+3)$	$(4K+2)$
s^1	$(-4K^2-12K+10)/(K+3)$	
s^0	$(4K+2)$	

pivot column

$$\Longrightarrow -4K^2-12K+10=0 \Longrightarrow \begin{cases} K = \cancel{-3.68} \\ K = 0.679 \end{cases}$$

The only value of $K > 0$ that makes elements of *pivot column* to vanish is $K \cong 0.68$

This "*Root Locus*" intercepts the *imaginary axis* in *2 points* $(\pm j\omega')$ for $K \cong 0.68$

Example 11 (*continued*) Application of Rule #8

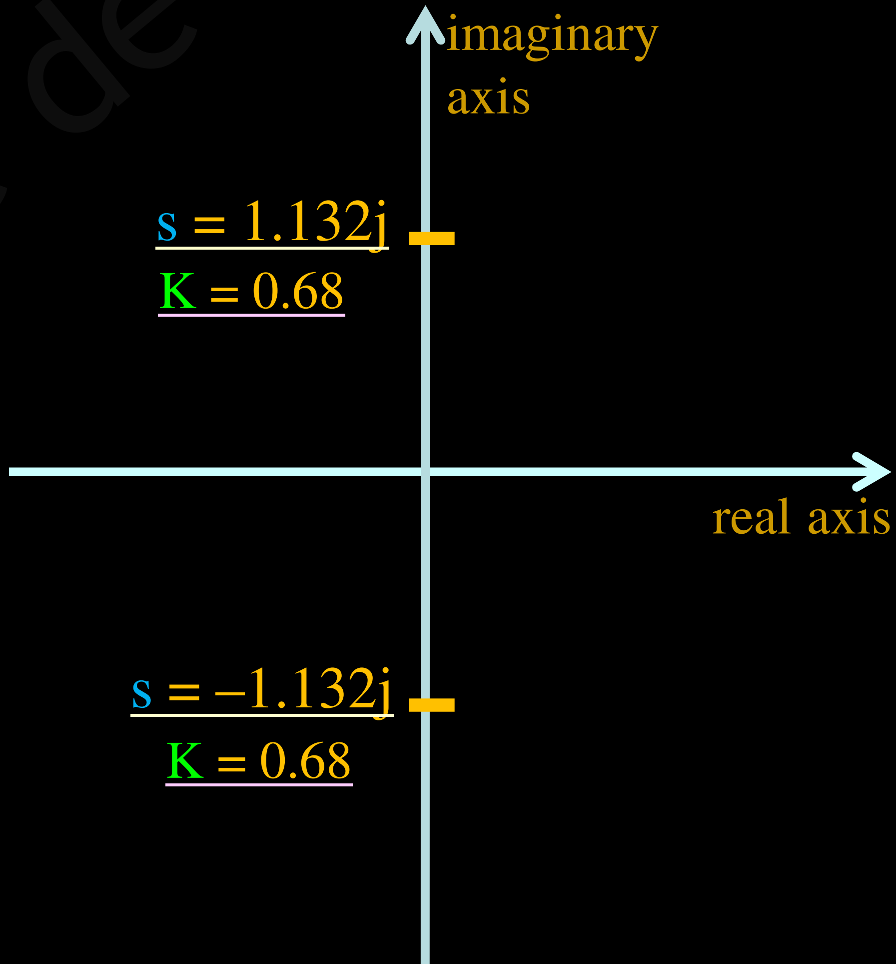
Now, in order to find these 2 values ($\pm j\omega'$) where the "*Root Locus*" intercept the *imaginary axis* when $K \cong 0.68$, we take the polynomial of the *line immediately above* (that is, the *line before the last*) and calculates the roots:

$$p(s) = (K+3)\,s^2 + (4K+2) = 0$$

$\Rightarrow$ $3.68\,s^2 + 4.72 = 0$

$\Rightarrow$ roots: $s = \pm\, 1.132j$

(thus, $\omega' = 1.132$)



Concluding, this "*Root Locus*" intercepts the *imaginary axis* in 2 points shown here in the graph

Examples

Sketch of the Root Locus
(Application of all the rules)

Example 12:

Sketching the complete "*Root Locus*" of the system of Example 4

$$G(s)H(s) = \frac{K \cdot (2s+1)}{(s-4)s}$$

$n = 2$

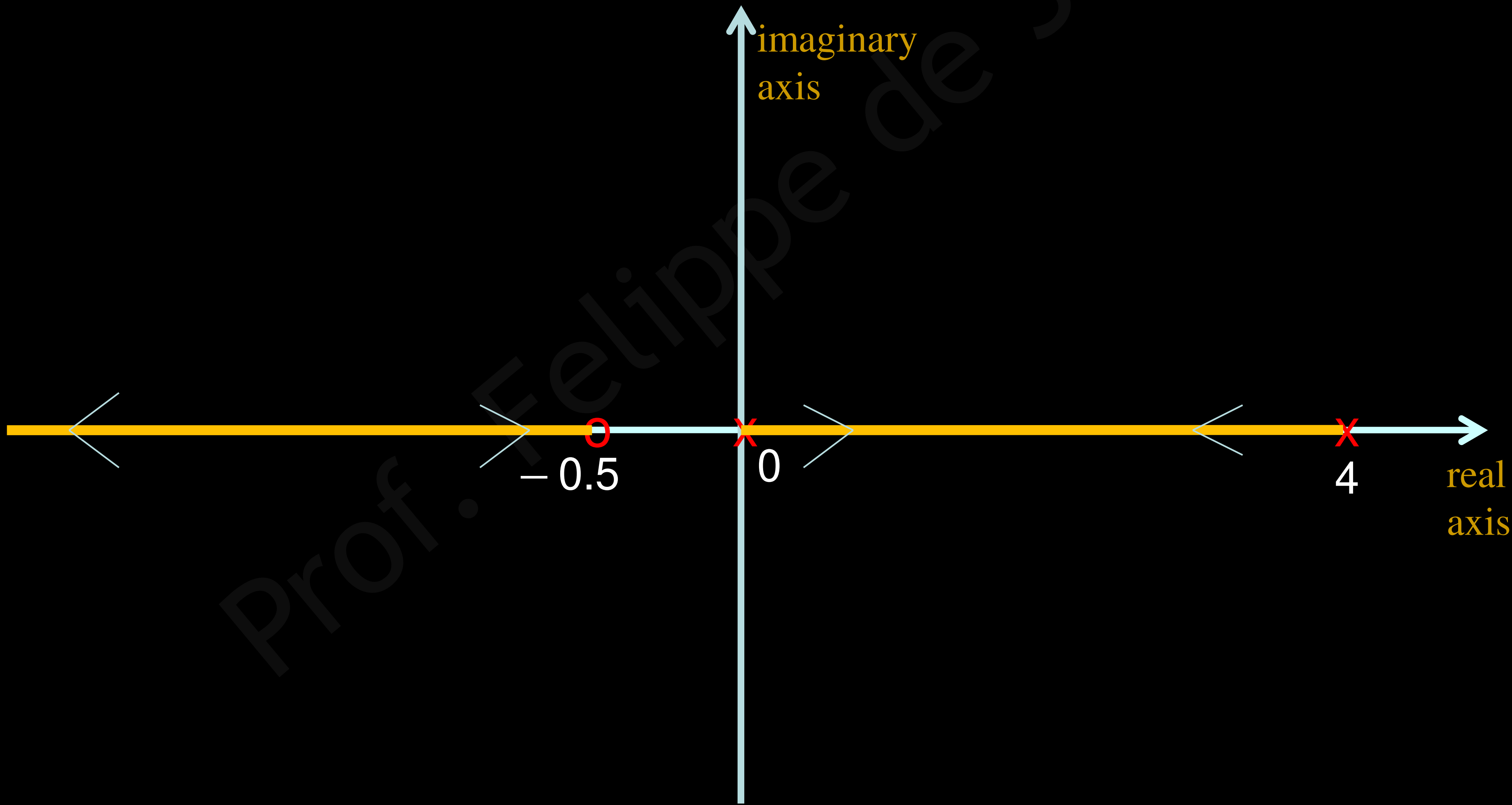
$m = 1$

This "*Root Locus*" has 2 *branches* (*Rule #1*)

This *C.L. system* has already appeared in Examples 1 and 6 (*part I*) and 10 (here in *part II*)

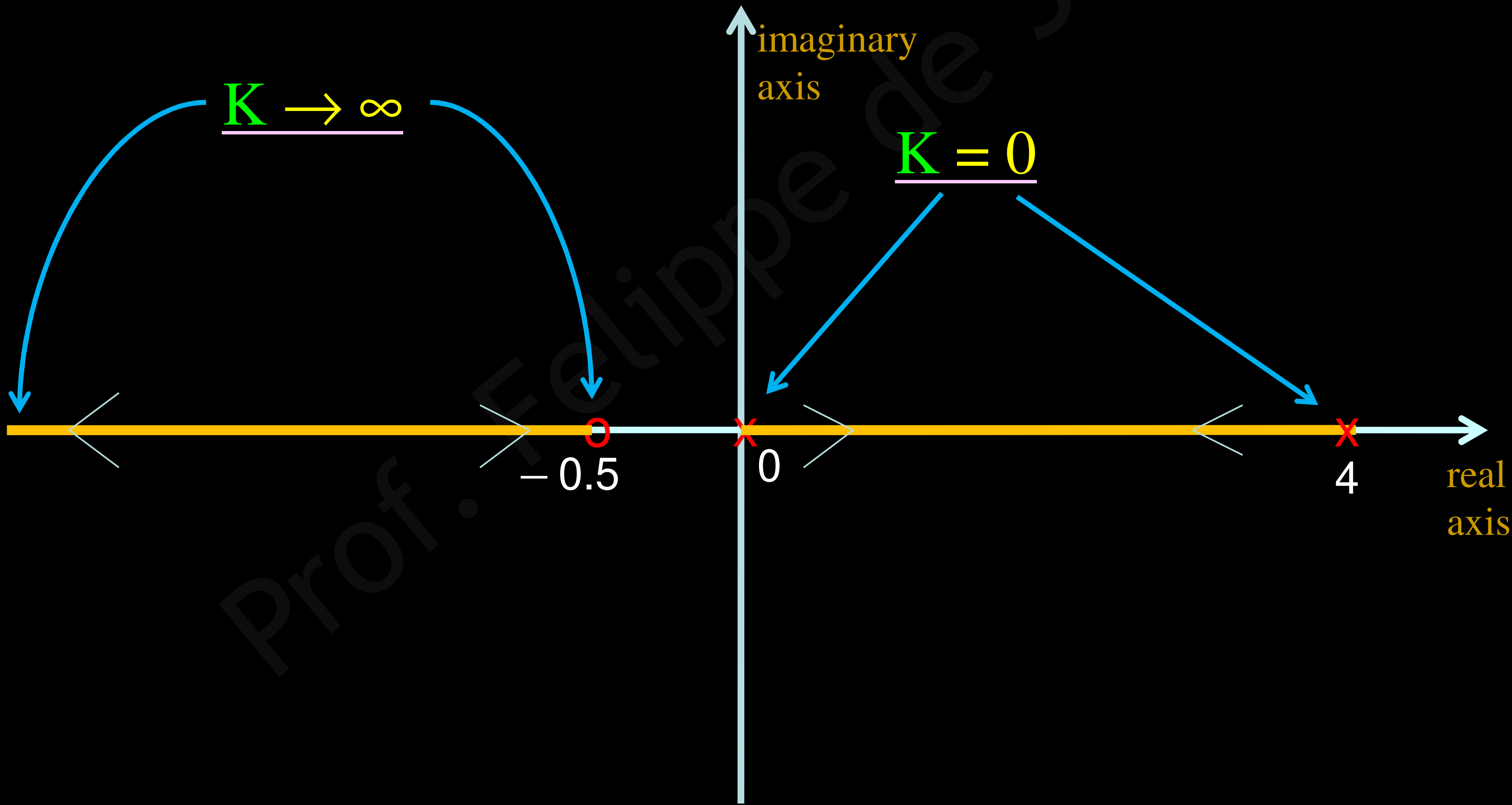
Example 12 (*continued*)

The intervals on the *real axis* (*Rule #2*), as well as the *points of beginning* and *ending* of the branches (*Rule #3*) are shown below.



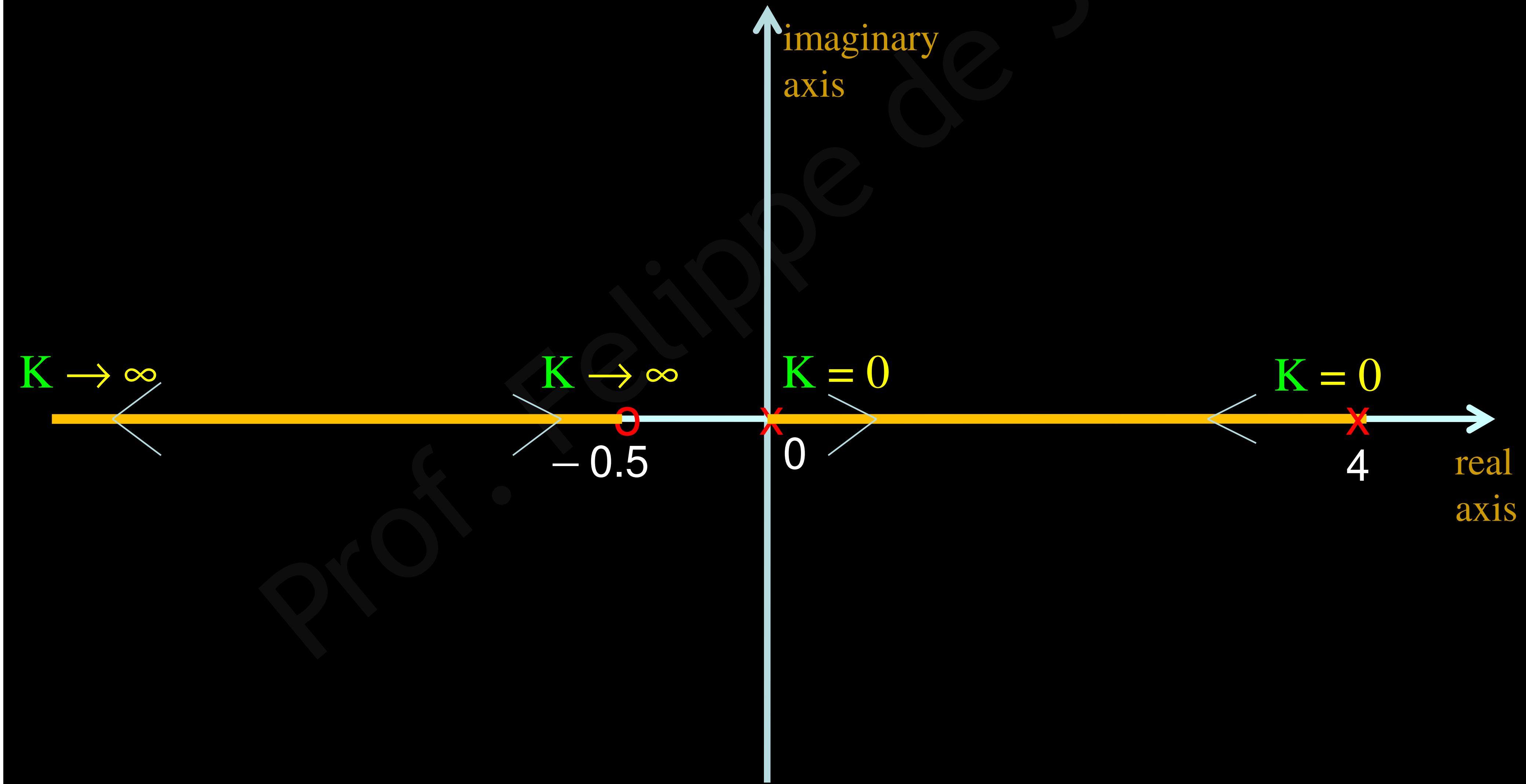
Example 12 (*continued*)

The intervals on the *real axis* (*Rule #2*), as well as the *points of beginning* and *ending* of the branches (*Rule #3*) are shown below.



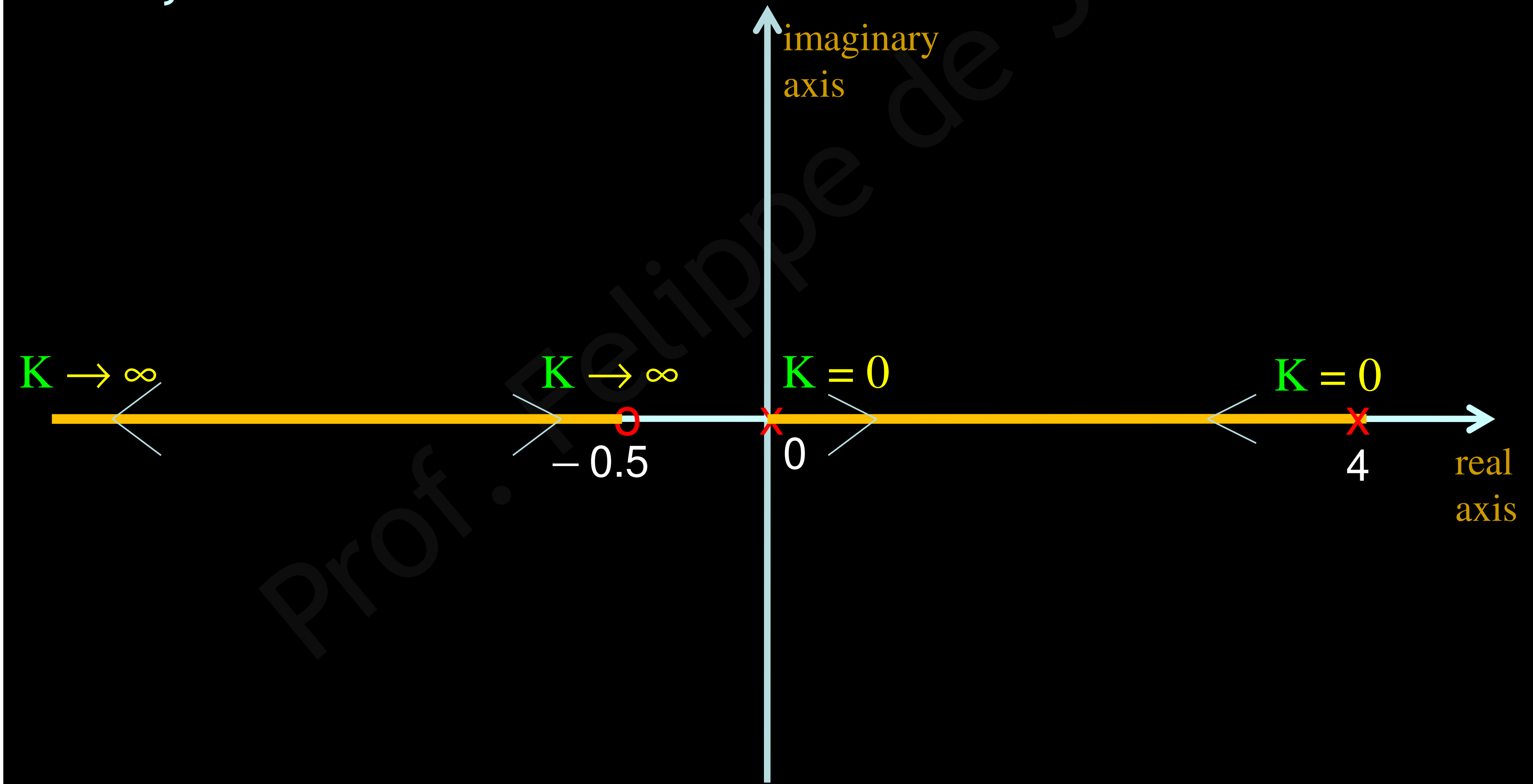
Example 12 (*continued*)

The intervals on the *real axis* (*Rule #2*), as well as the *points of beginning* and *ending* of the branches (*Rule #3*) are shown below.



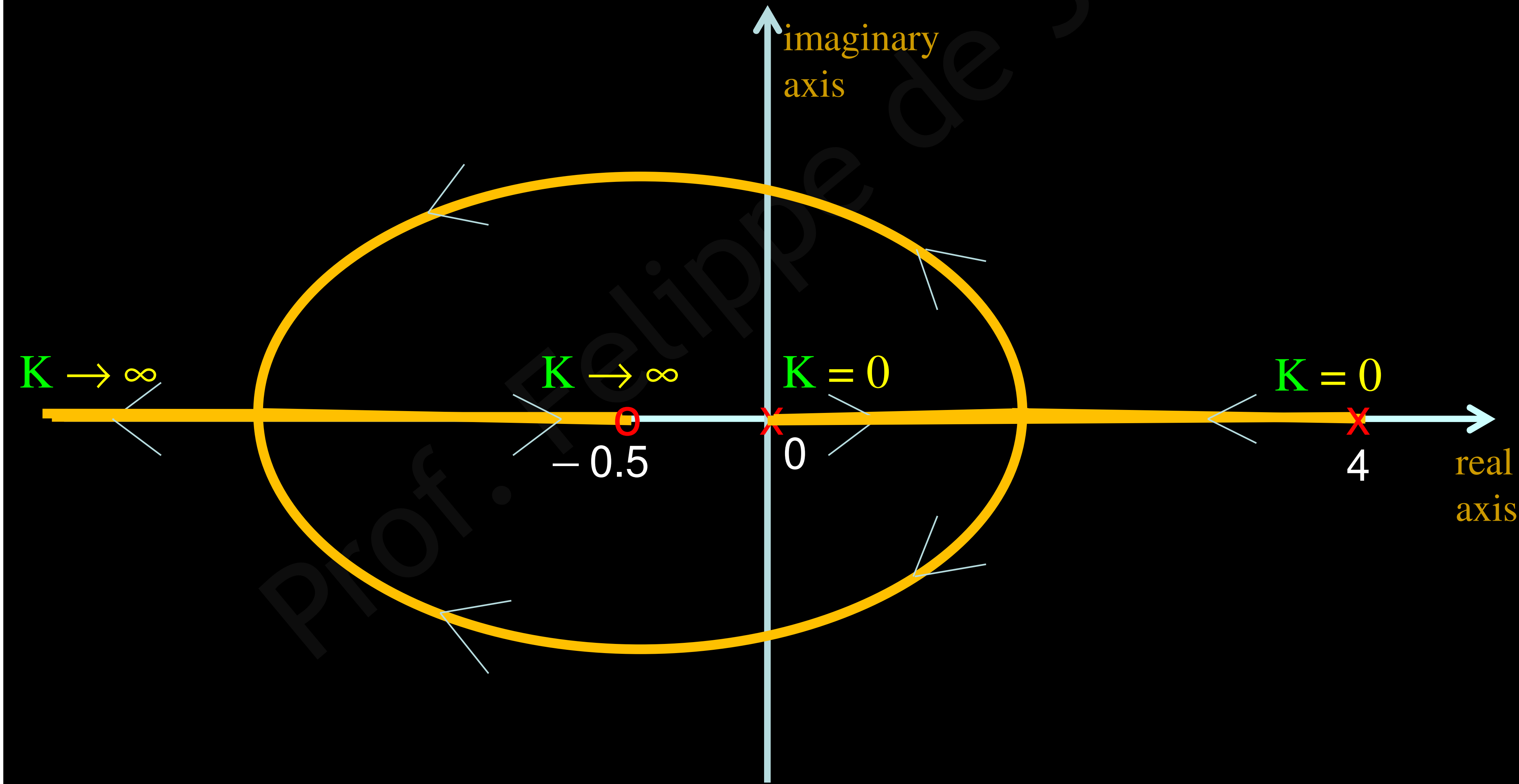
Example 12 (*continued*)

The only *asymptote* at the infinite occurs in $\gamma = 180°$ (*Rule #4*).
The meeting point of the *asymptote* $\sigma_o = 4.5$ (*Rule #5*), although in this case it is not necessary since the *Root Locus* lies entirely in the *real axis*.



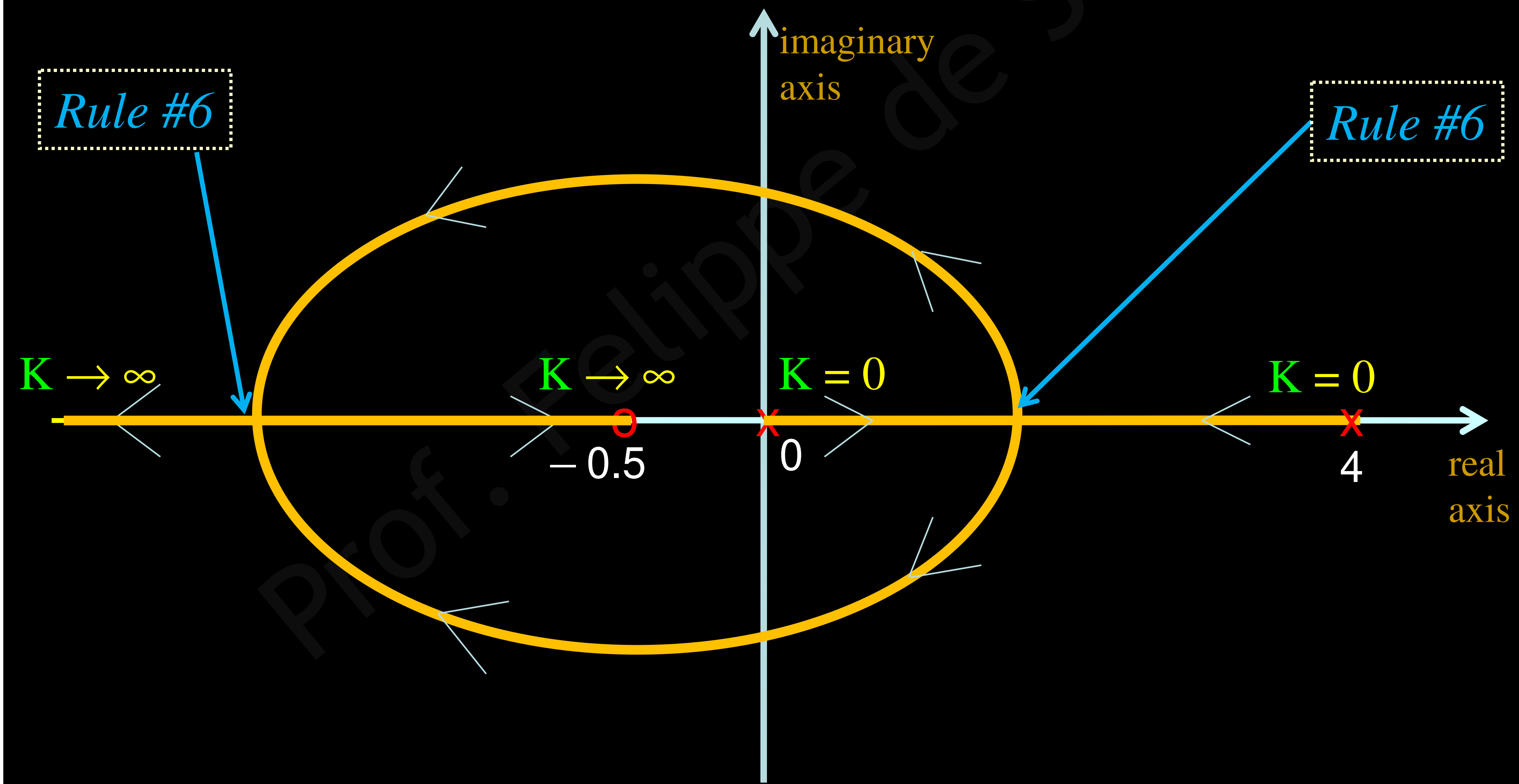
Example 12 (*continued*)

It is already possible to predict that 2 *branches* meet on the *right* and LEAVE the *real axis* to meet again on the *left* when they ENTER the *real axis* again.



Example 12 (*continued*)

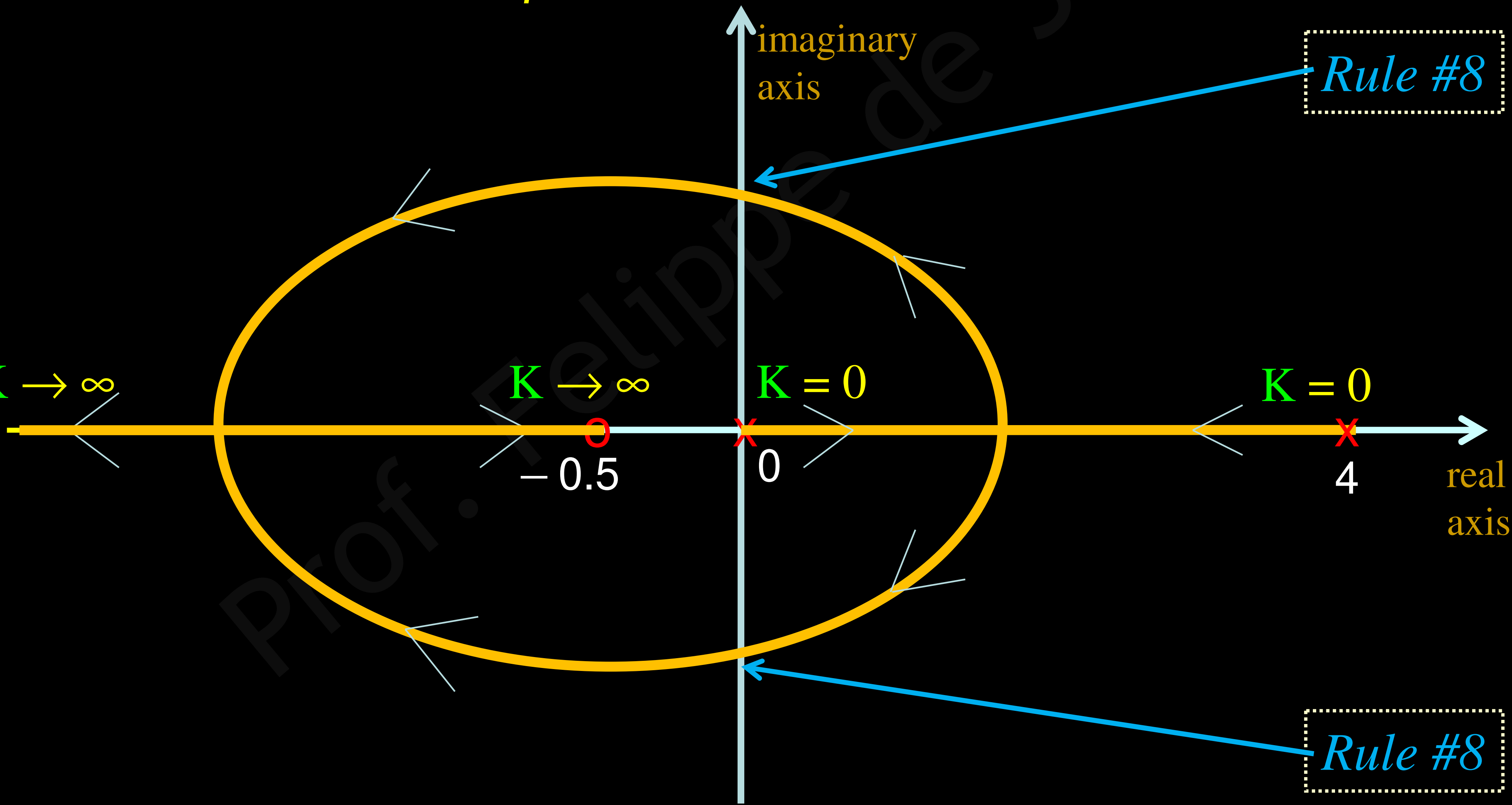
However, only by ***Rule #6*** it will be possible to exactly determine what these *points* are



Example 12 (*continued*)

On the other hand, it is already possible to predict that the 2 *branches* intercept the *imaginary axis*

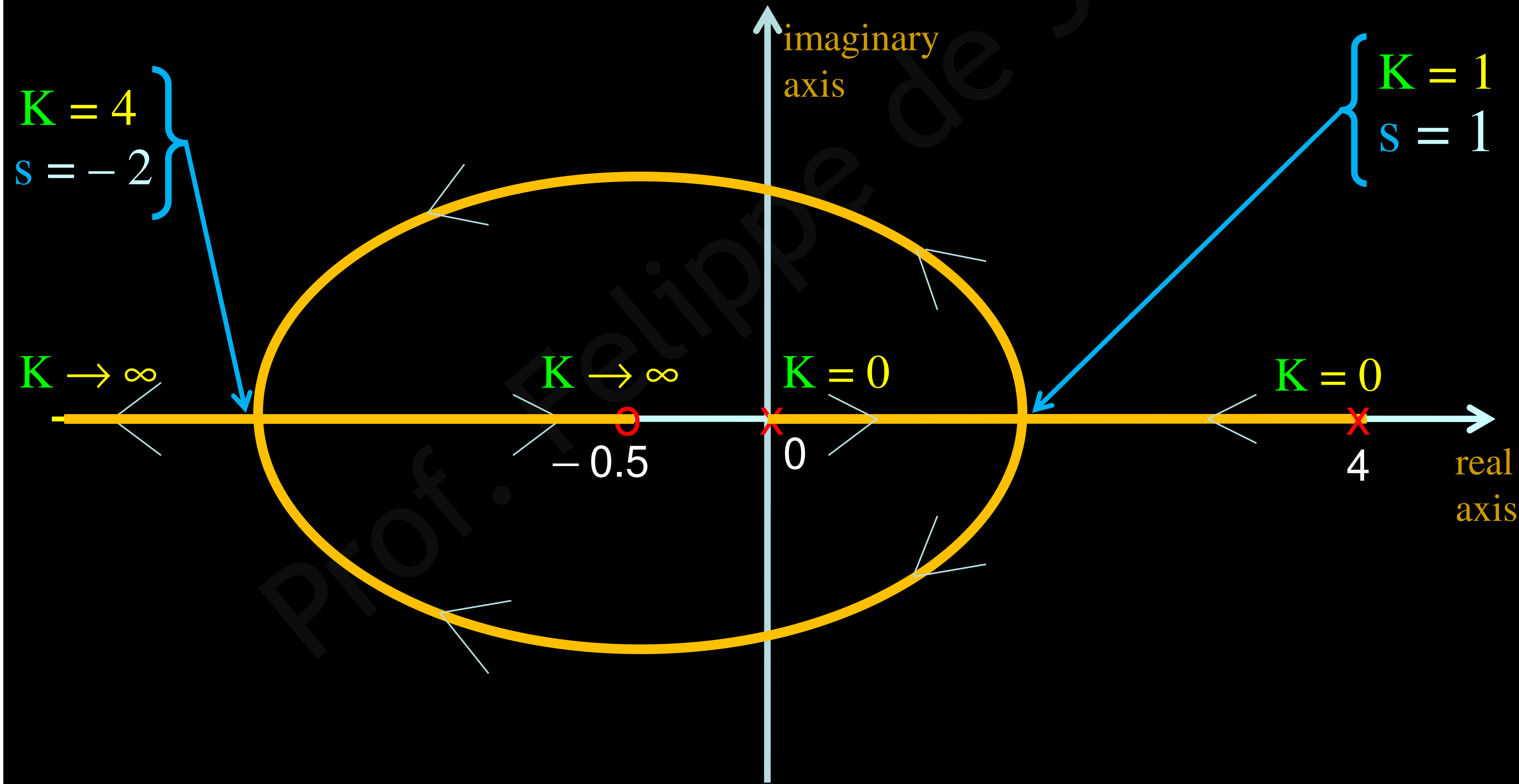
However, only by *Rule #8* it will be possible to exactly determine where these *interceptions* occur



Example 12 (*continued*)

As we have seen (Example 6, parte 1), the *points* where branches meet in the *real axis* (*Rule #6*) are:

$s = 1$ (for $K = 1$) and $s = -2$ (for $K = 4$)



Example 12 (*continued*)

As we have seen (Example 6, parte 1), the *points* where branches meet in the real axis (*Rule #6*) are:

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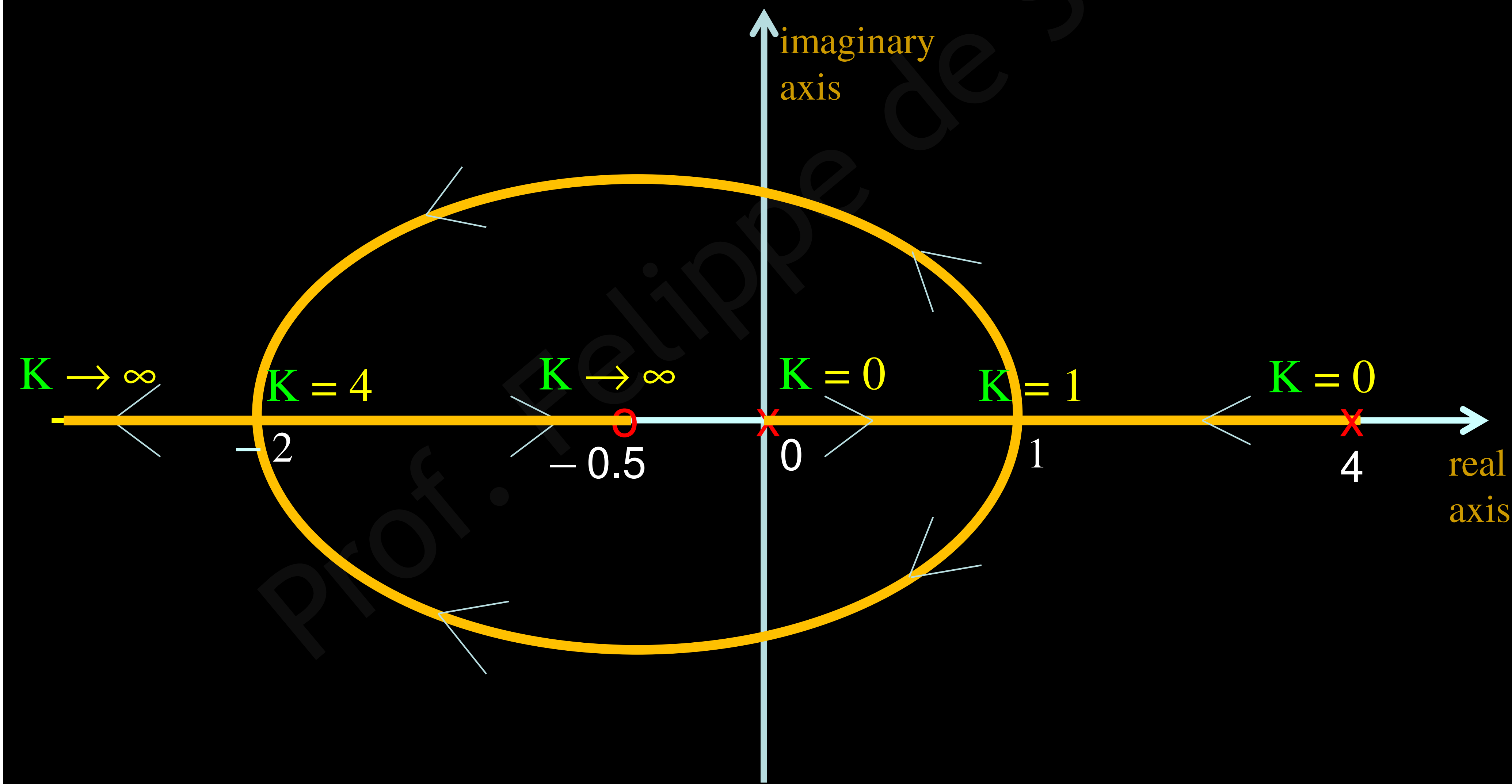
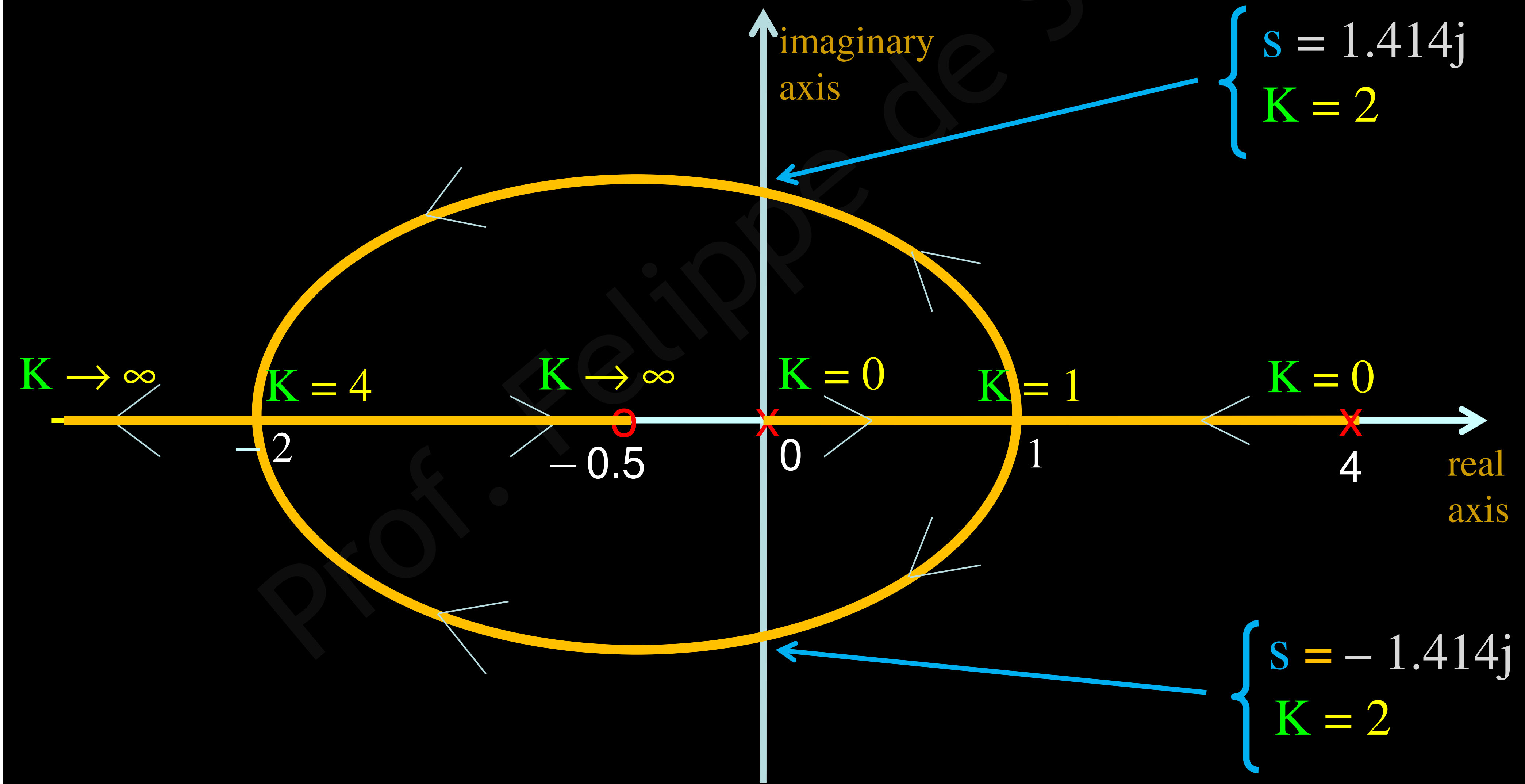


Example 12 (*continued*)

As we have seen (Example 10), the *meeting points* with *imaginary axis* (*Rule #8*) are:

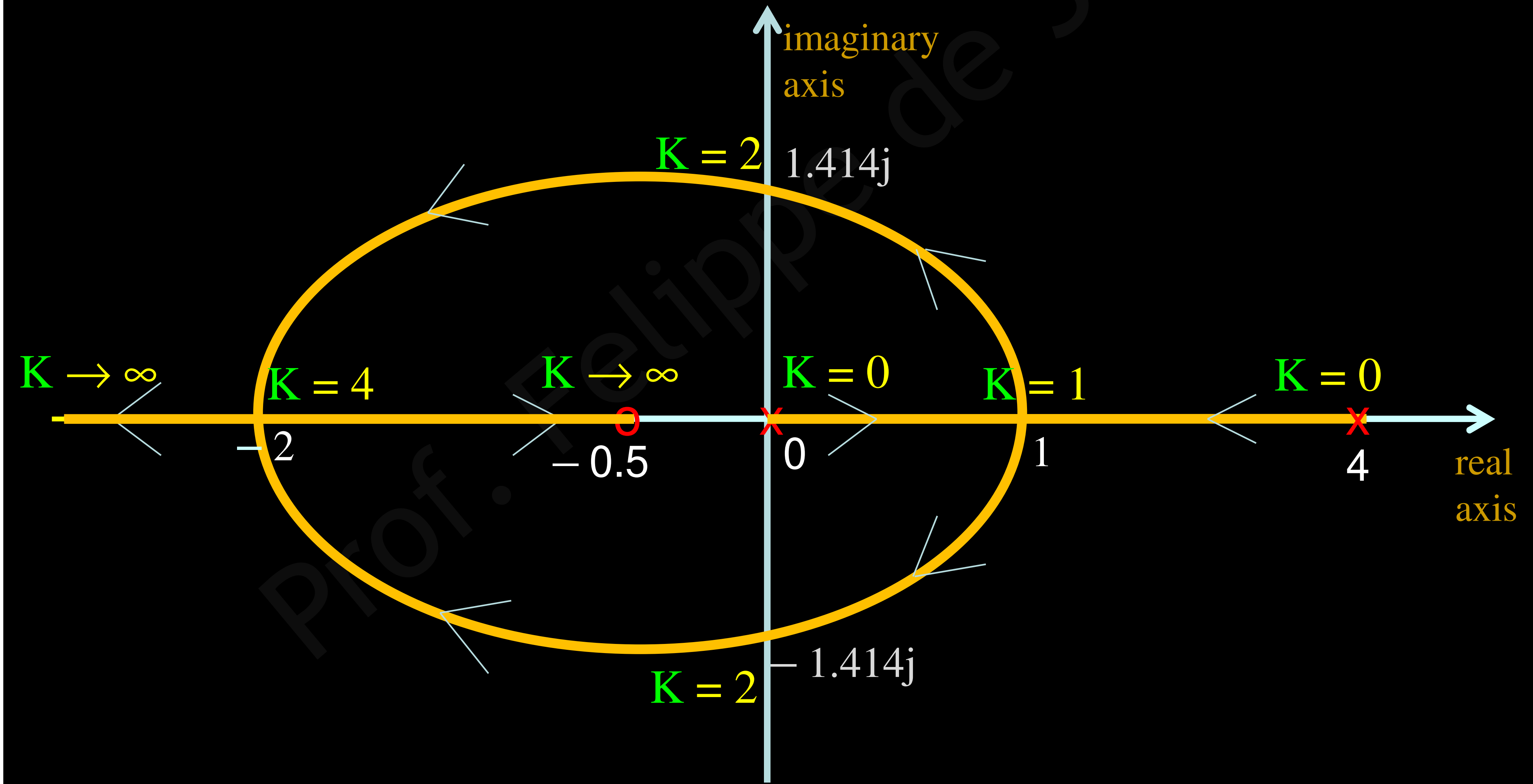
$$s = \pm 1.414j \quad (\text{for } K = 2)$$



Example 12 (*continued*)

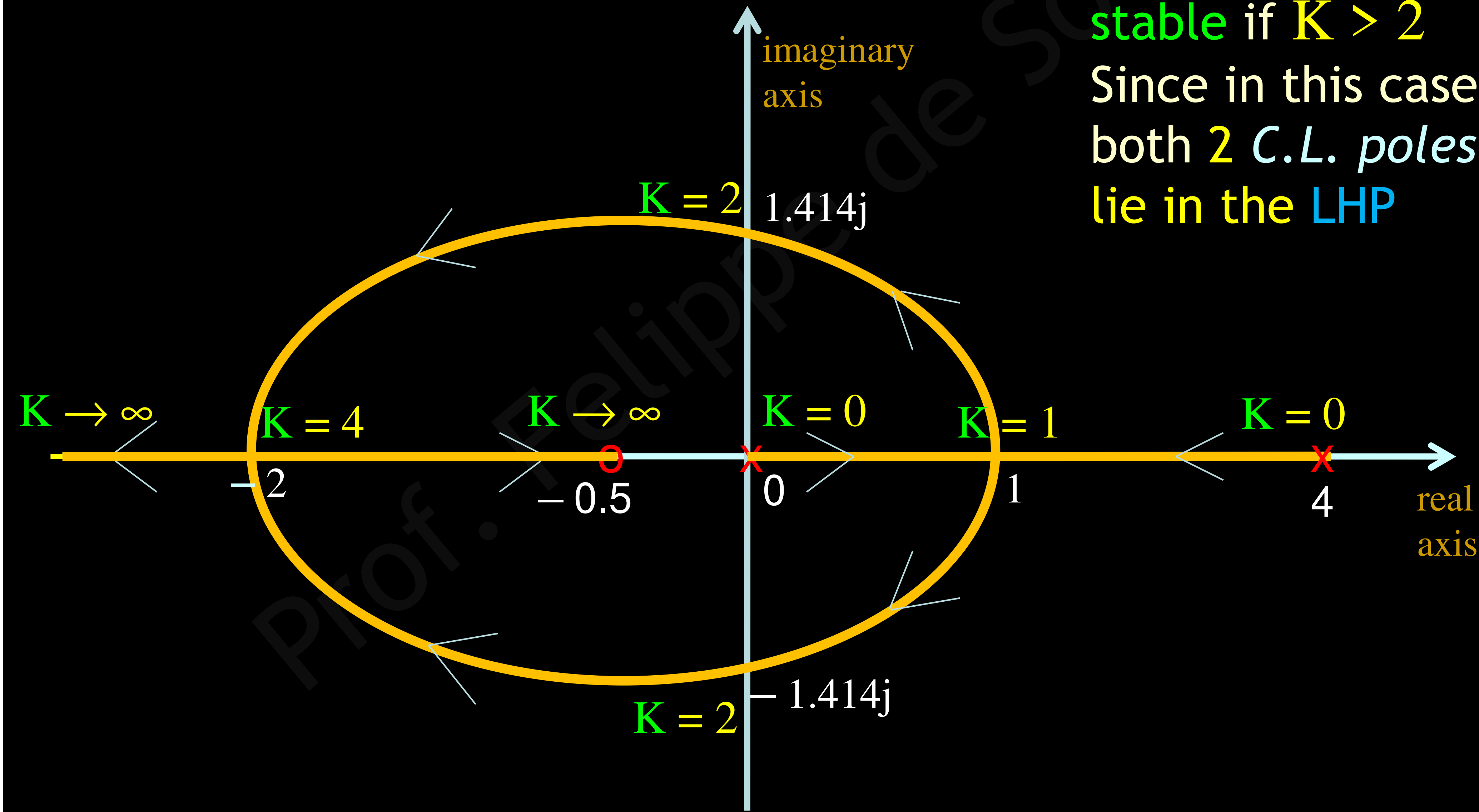
As we have seen (Example 10), the *meeting points* with *imaginary axis* (*Rule #8*) are:

$s = \pm 1.414j$ (for $K = 2$)



Example 12 (*continued*)

and the "*Root Locus*" now is complete.



Observe that the *C.L. system* is stable if $K > 2$

Since in this case both 2 *C.L. poles* lie in the LHP

Example 13:

Sketching the "*Root Locus*" for

$$G(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

$n = 3$

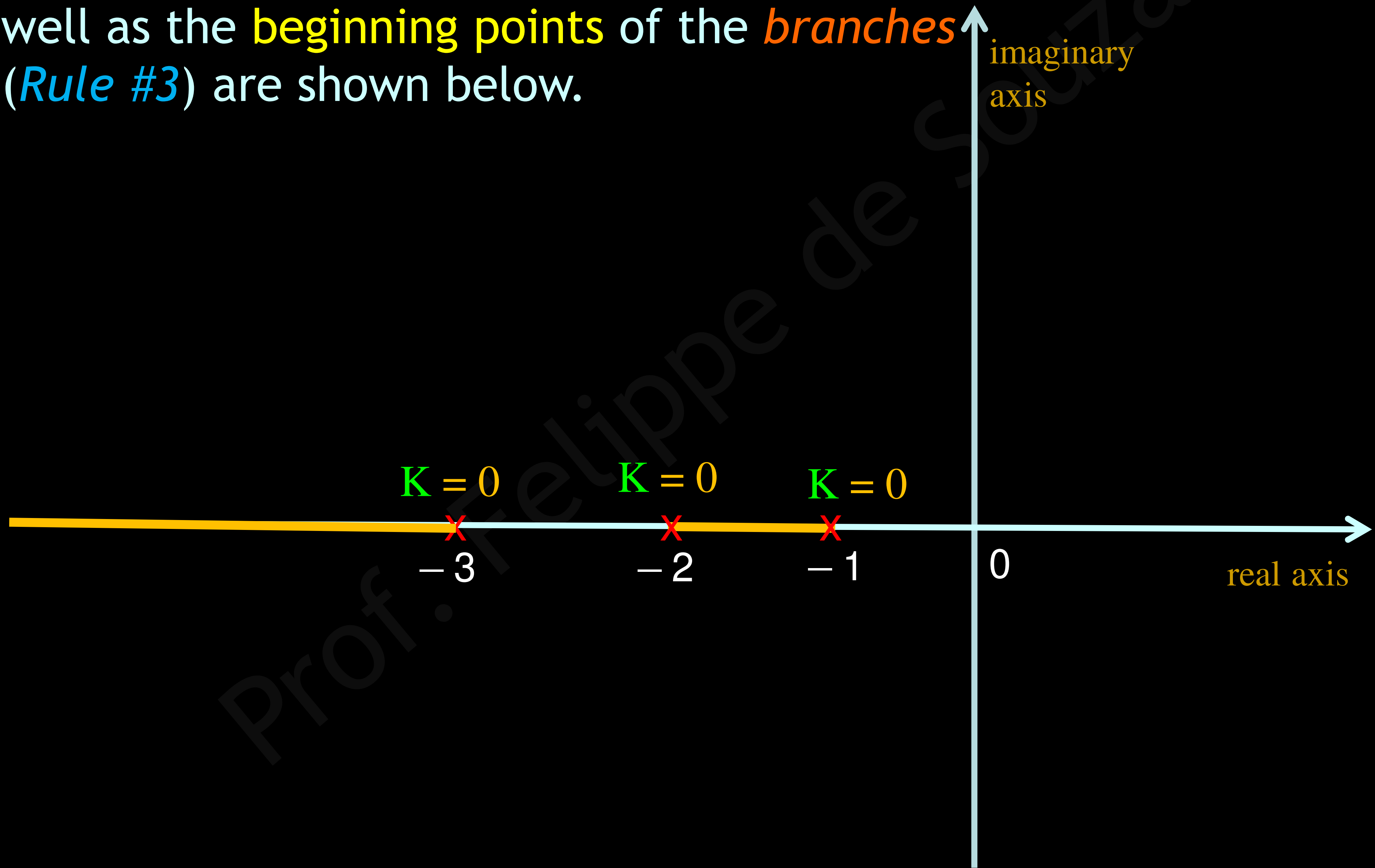
$m = 0$

This "*Root Locus*" has 3 *branches* (*Rule #1*)

We have already seen this *C.L. system* in Examples 5 and 7 (part I)

Example 13 (*continued*)

The intervals in the *real axis* (*Rule #2*), as well as the beginning points of the *branches* (*Rule #3*) are shown below.

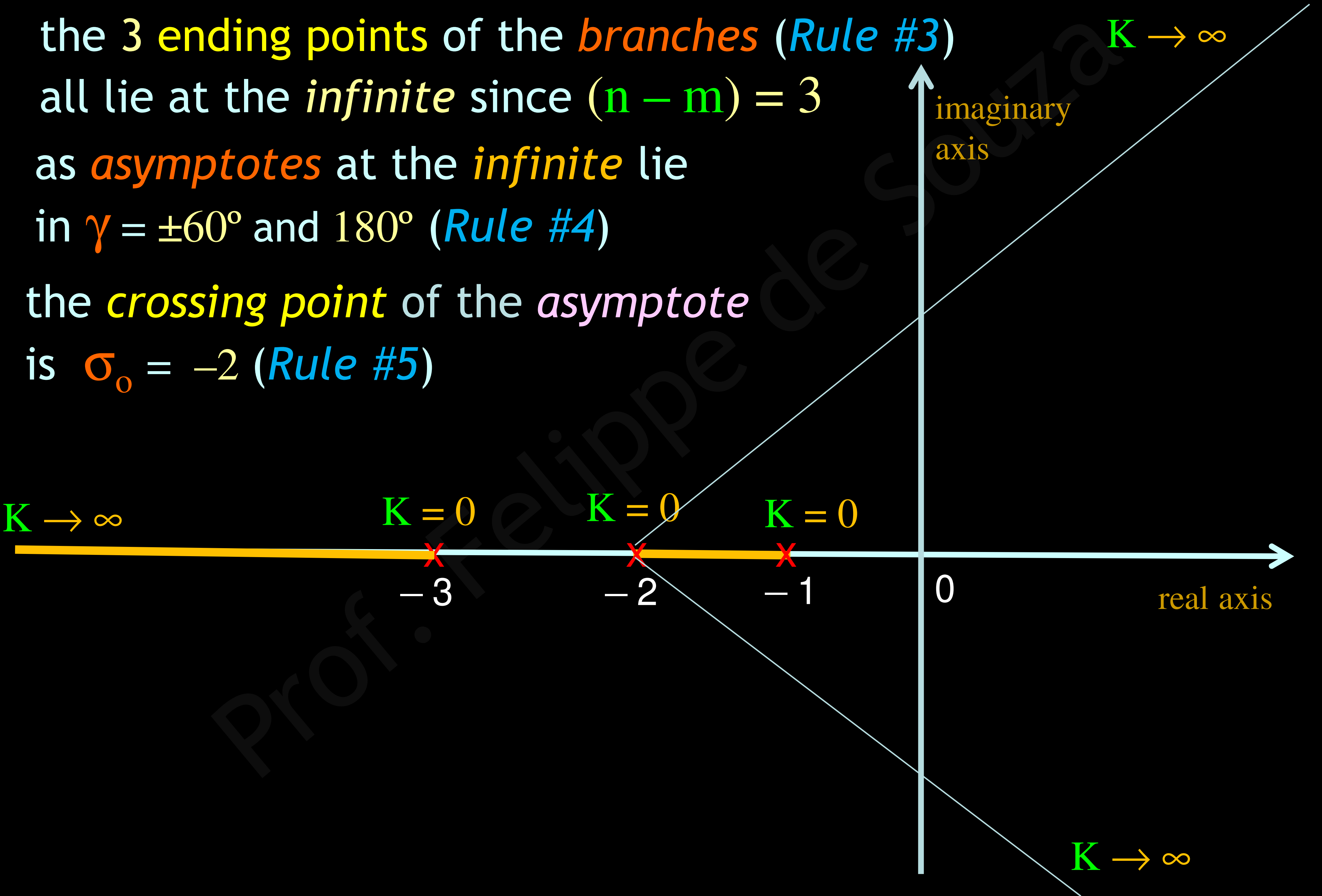


Example 13 (*continued*)

the 3 ending points of the *branches* (*Rule #3*)
all lie at the *infinite* since $(n - m) = 3$

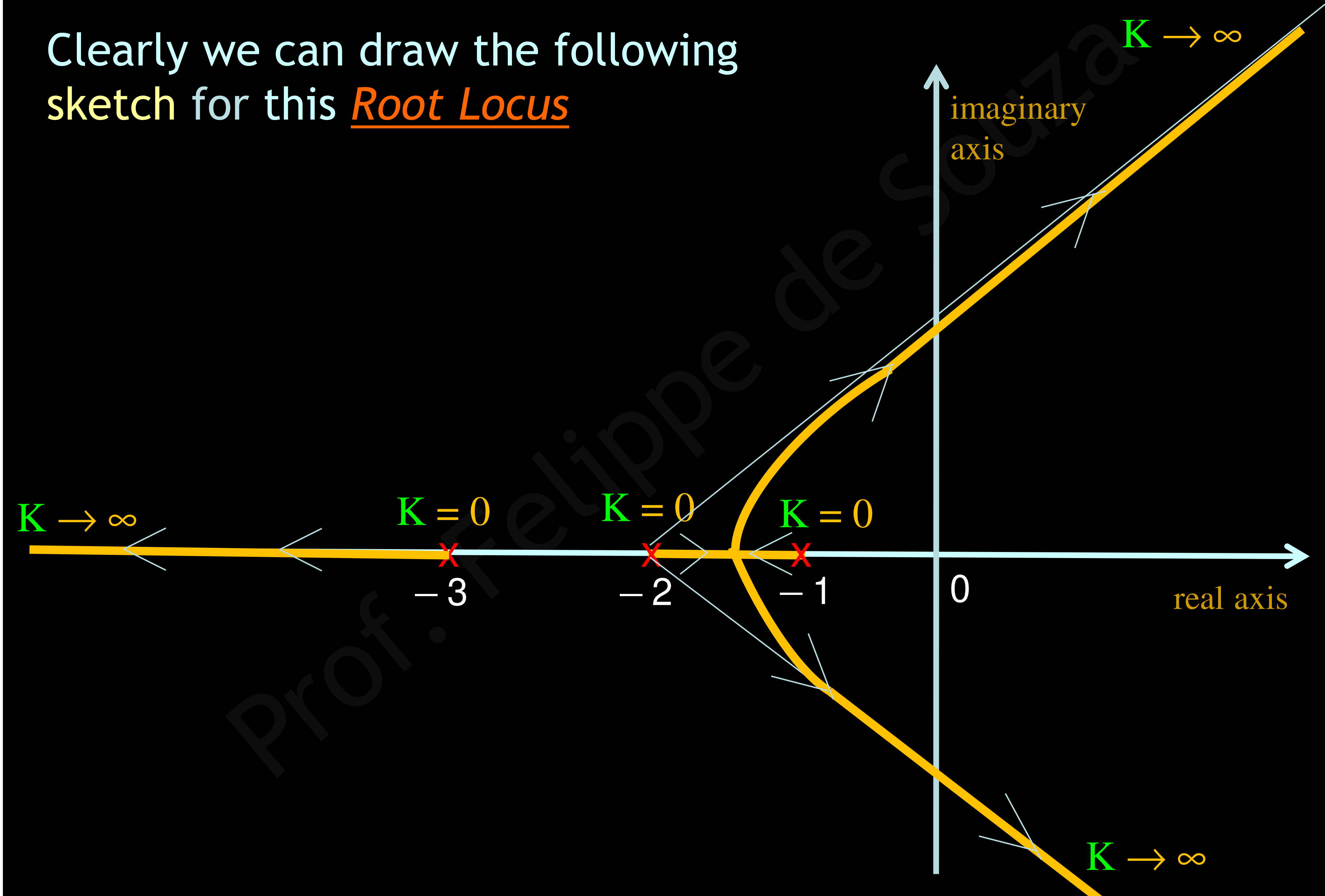
as *asymptotes* at the *infinite* lie
in $\gamma = \pm 60^o$ and 180^o (*Rule #4*)

the *crossing point* of the *asymptote*
is $\sigma_o = -2$ (*Rule #5*)



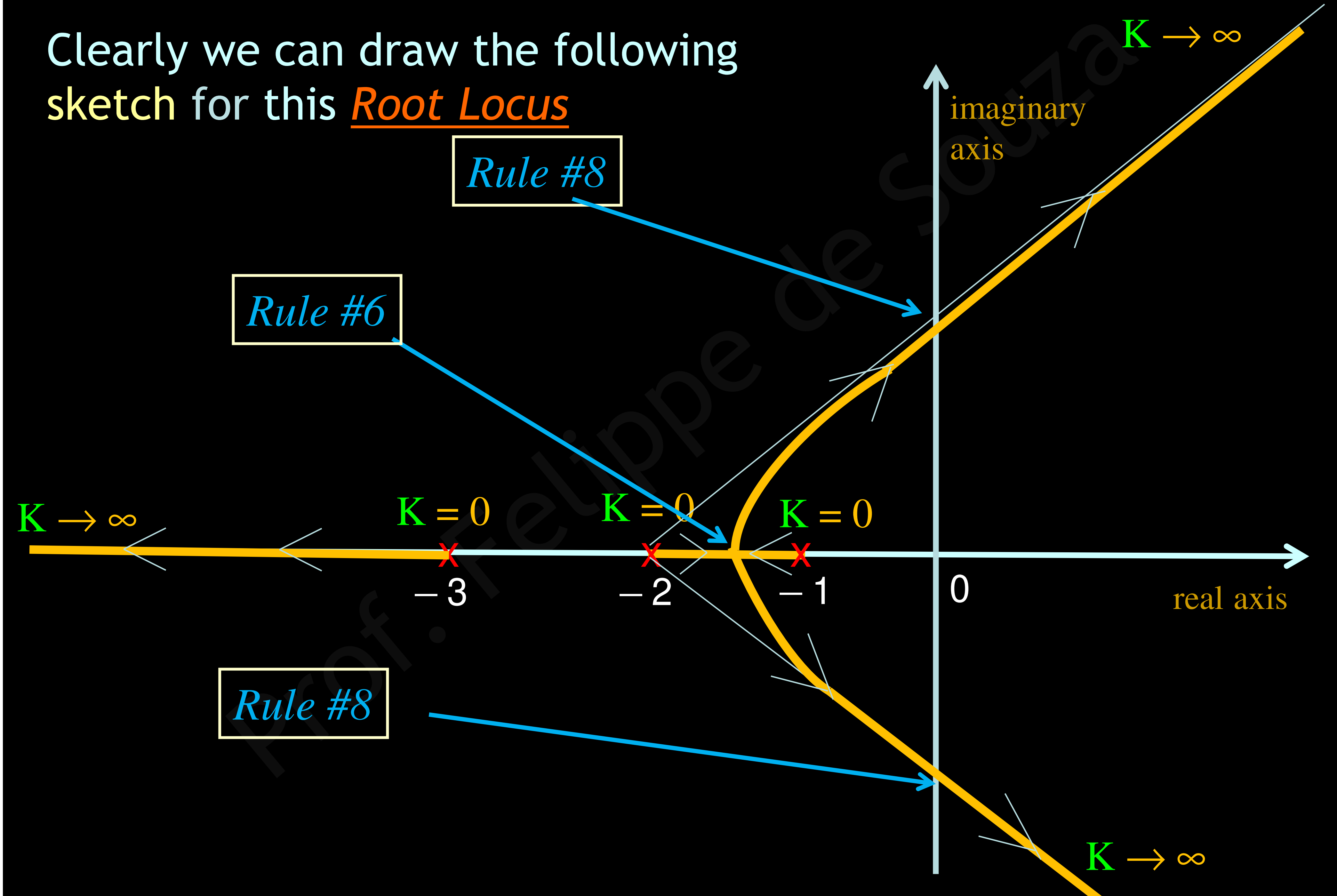
Example 13 (*continued*)

Clearly we can draw the following sketch for this *Root Locus*



Example 13 (*continued*)

Clearly we can draw the following sketch for this *Root Locus*

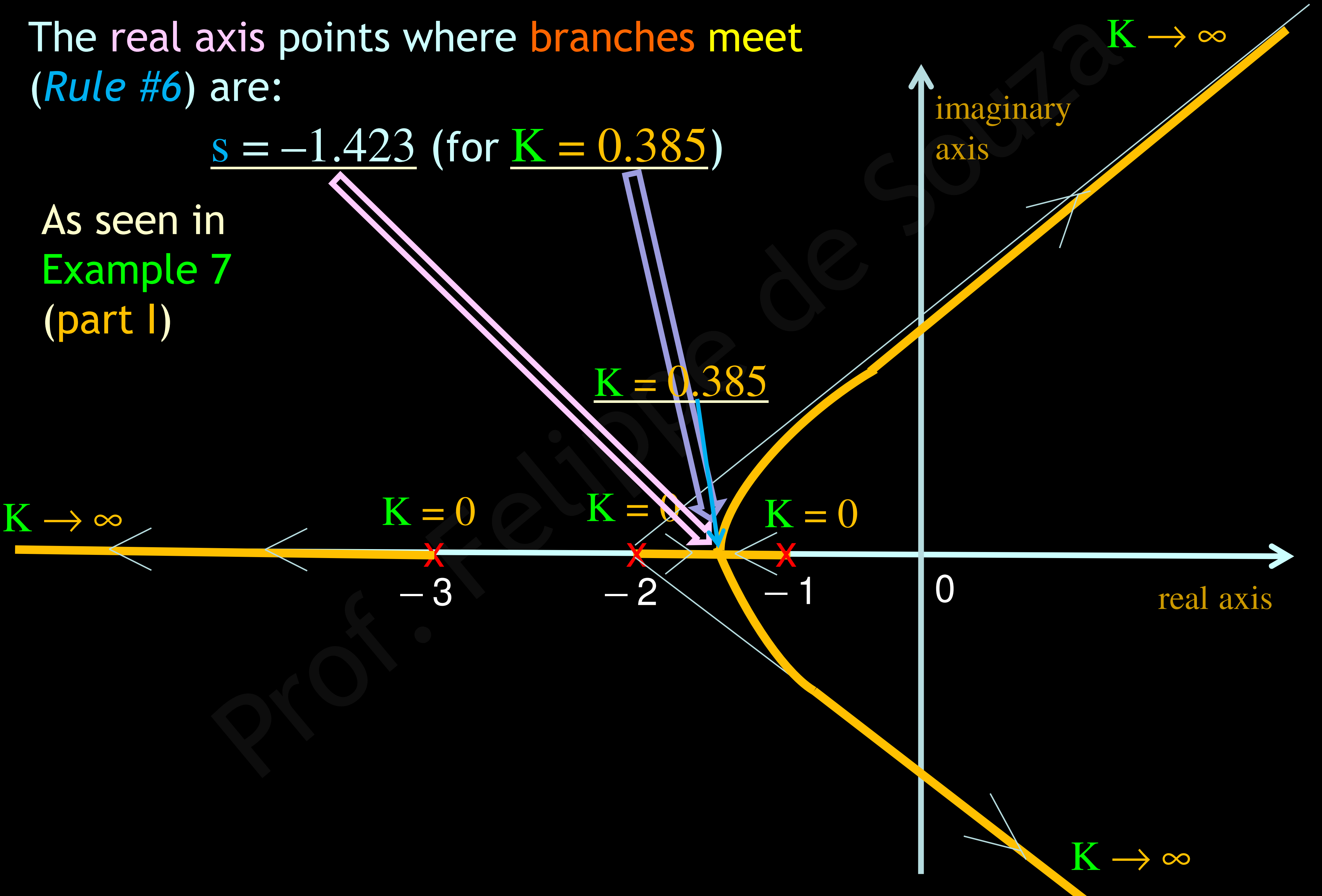


Example 13 (*continued*)

The real axis points where branches meet (*Rule #6*) are:

$s = -1.423$ (for $K = 0.385$)

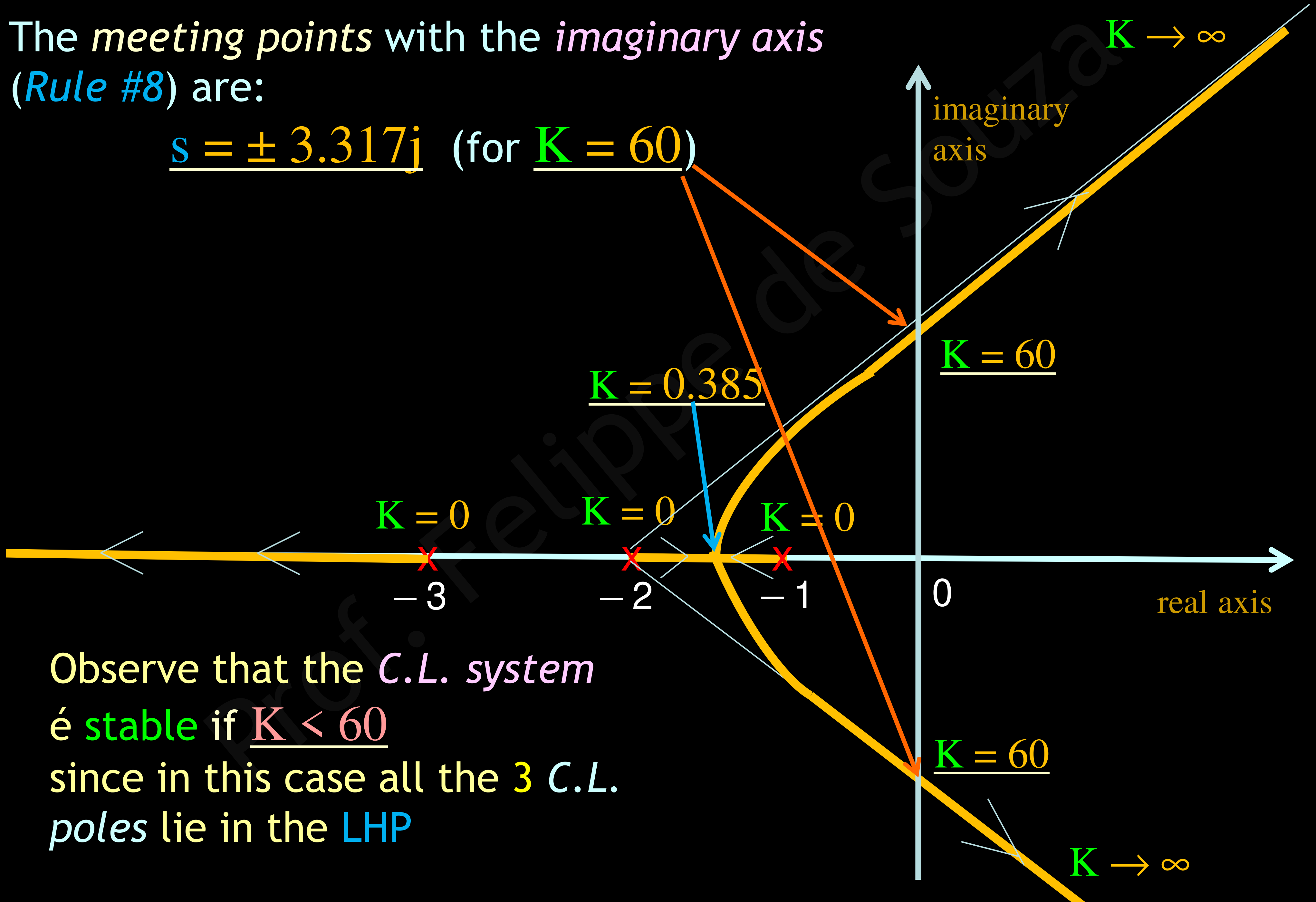
As seen in Example 7 (part I)



Example 13 (*continued*)

The *meeting points* with the *imaginary axis* (*Rule #8*) are:

$$s = \pm\, 3.317j \quad (\text{for } K = 60)$$



Observe that the *C.L. system* é stable if $K < 60$ since in this case all the 3 *C.L. poles* lie in the LHP

Example 14:

Sketching the "*Root Locus*" for

$$G(s)H(s) = \frac{K \cdot (s-2)^2}{(s^2+2s+2) \cdot (s+1)}$$

$n = 3$

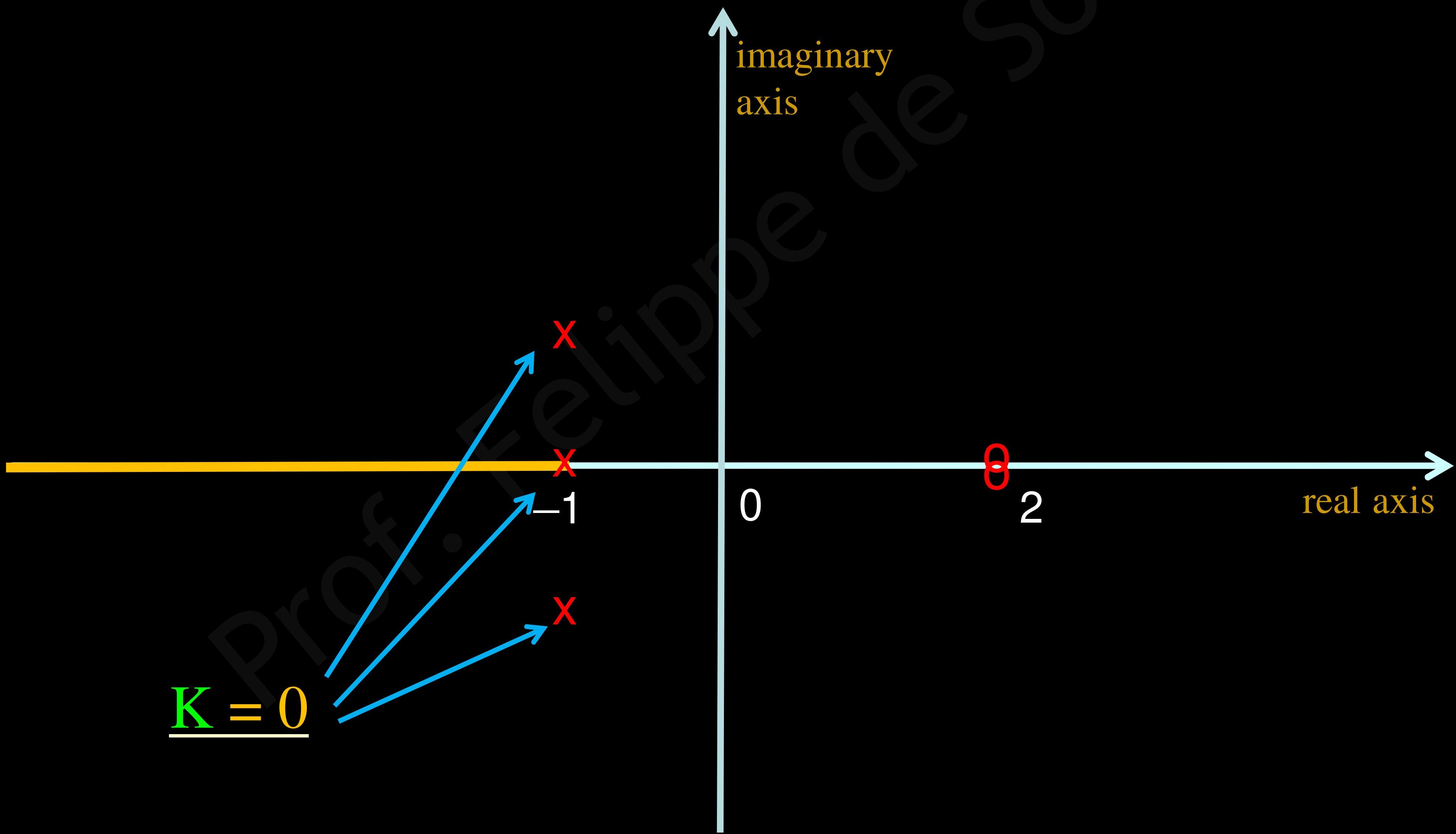
$m = 2$

This "*Root Locus*" has 3 *branches* (*Rule #1*)

We have already seen this *C.L. system* in Example 11 (here in part II)

Example 14 (*continued*)

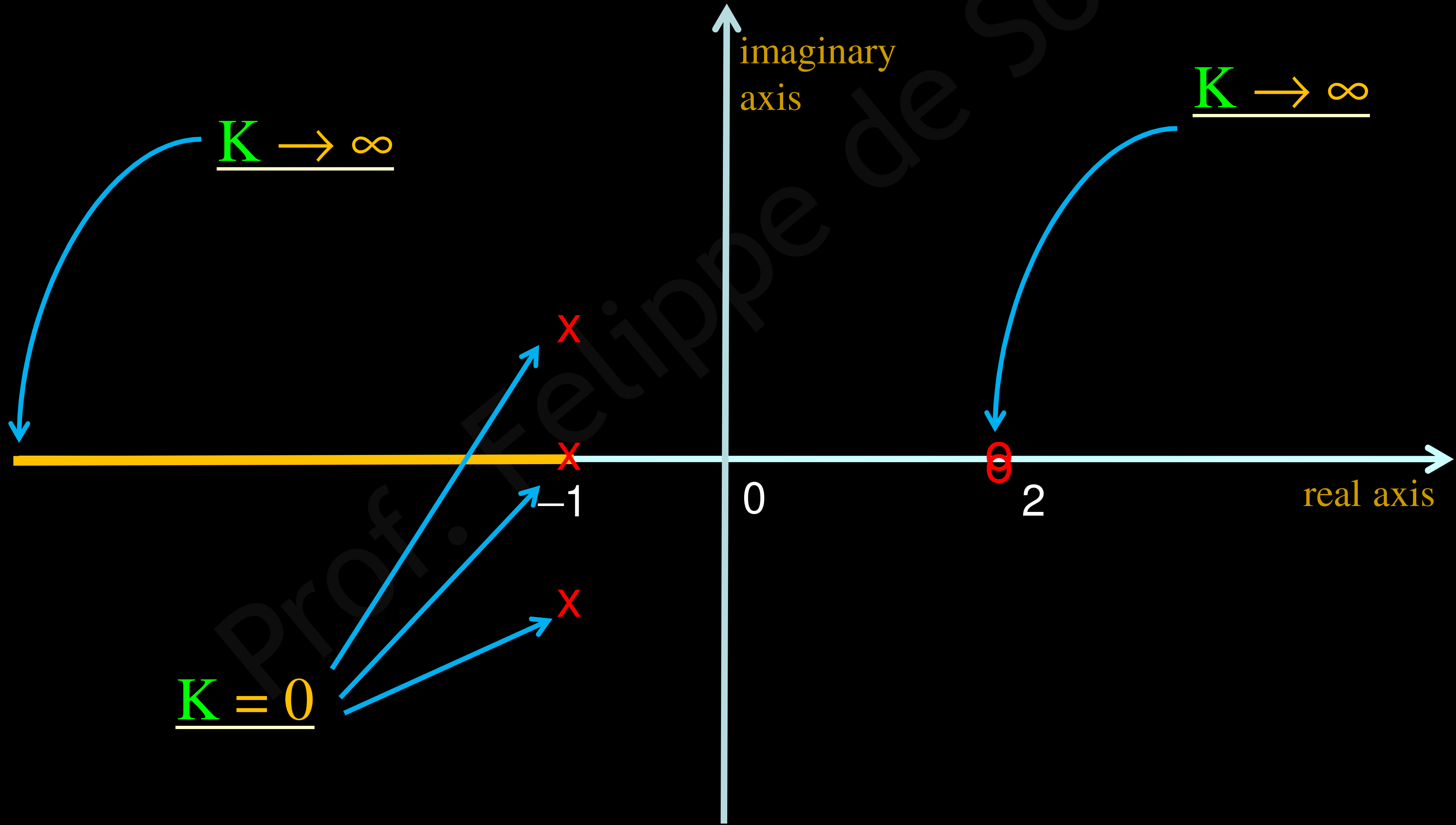
The intervals on the *real axis* (*Rule #2*), as well as beginning and ending *points* of the *branches* (*Rule #3*) are shown below



Example 14 (*continued*)

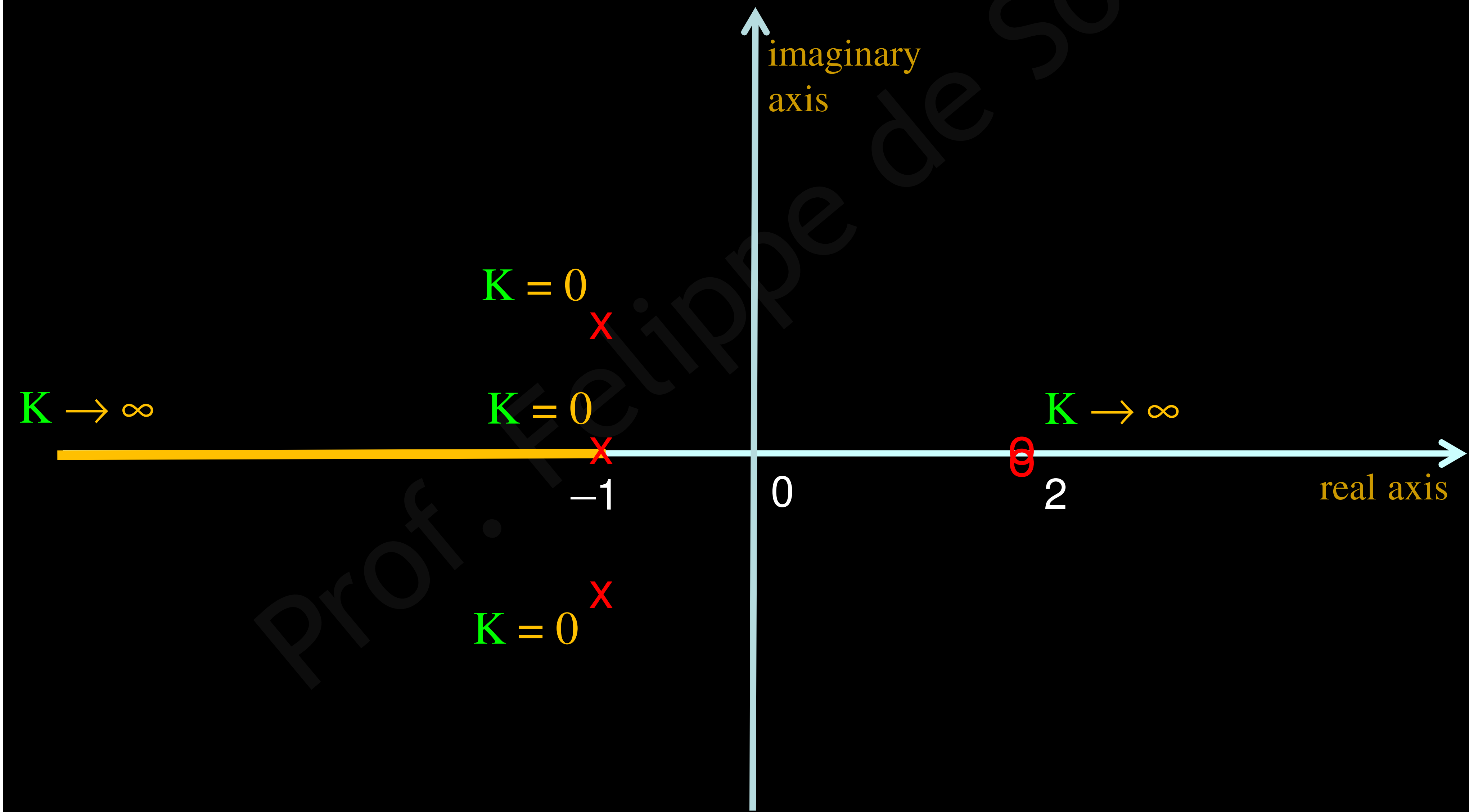
The only *asymptote* at the infinite is at $\gamma = 180°$ (*Rule #4*)

The crossing *point* of the *asymptote* $\sigma_o = -7$ (*Rule #5*), although on this case it was not necessary



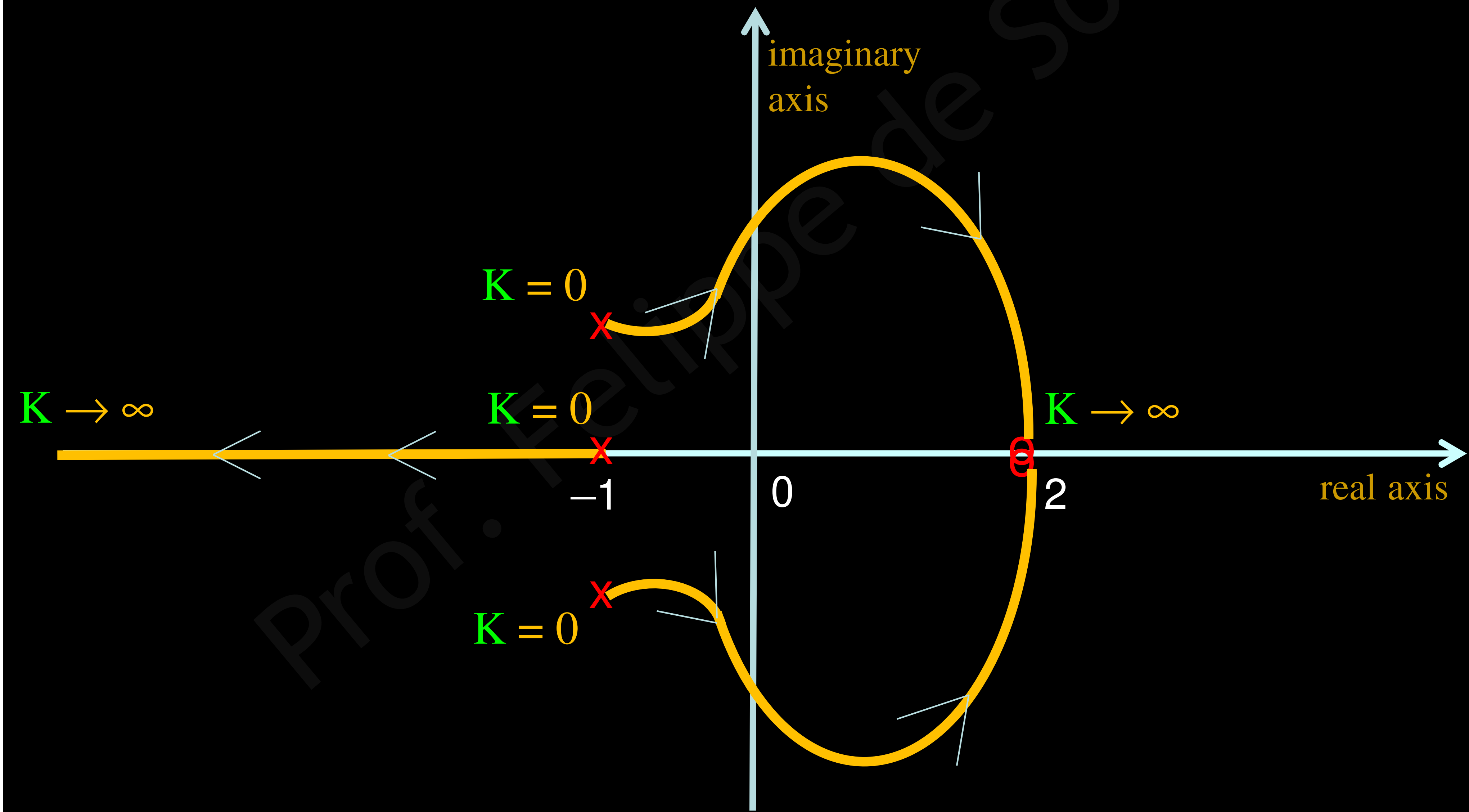
Example 14 (*continued*)

Summarizing, the intervals in the *real axis* (*Rule #2*), as well as the beginning and ending *points* of the *branches* (*Rule #3*) are shown below



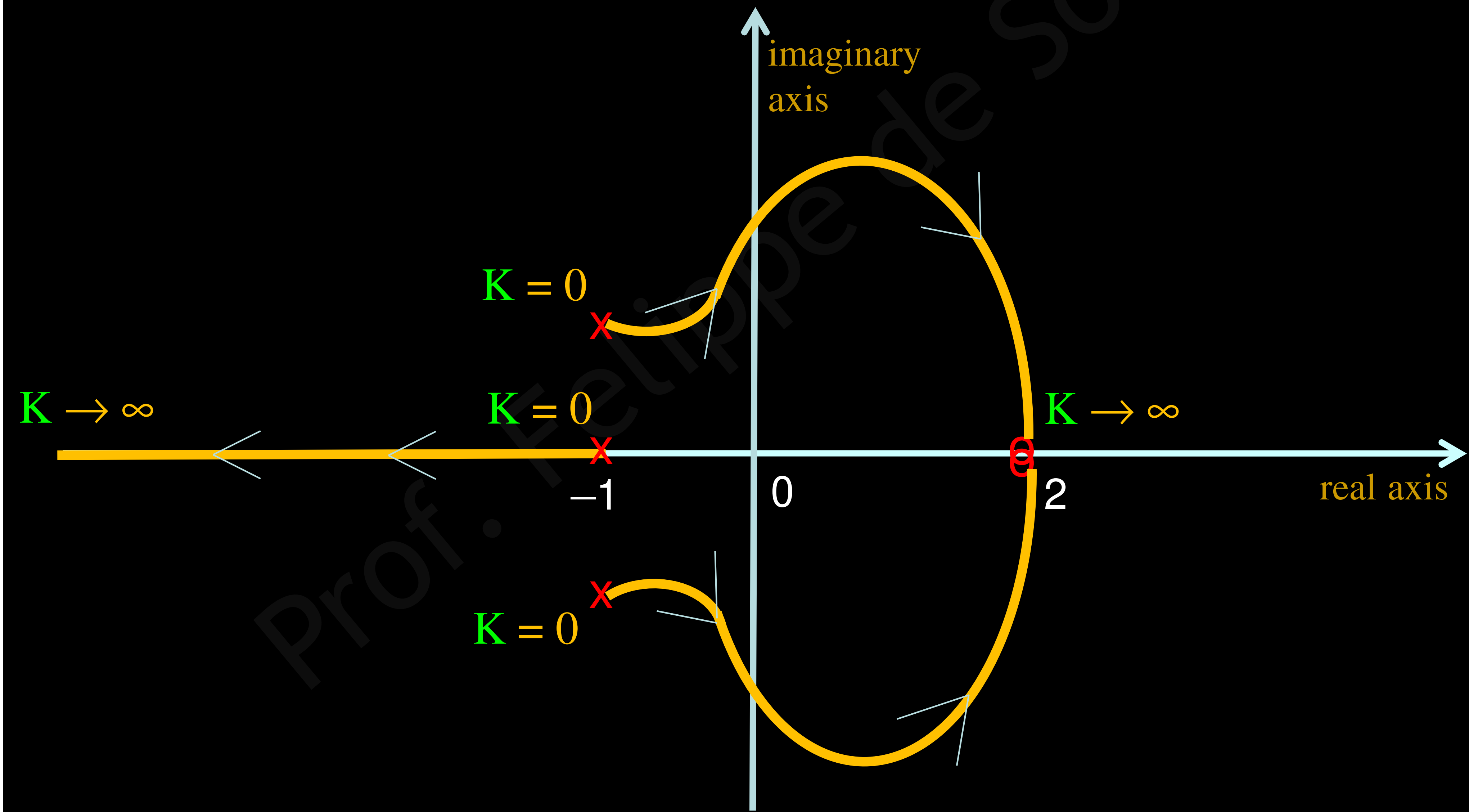
Example 14 (*continued*)

We ca predict that 2 *branches* that start at the 2 *complex* poles $s = -1 \pm j$ go to the right to meet the double zeros at $s = 2$



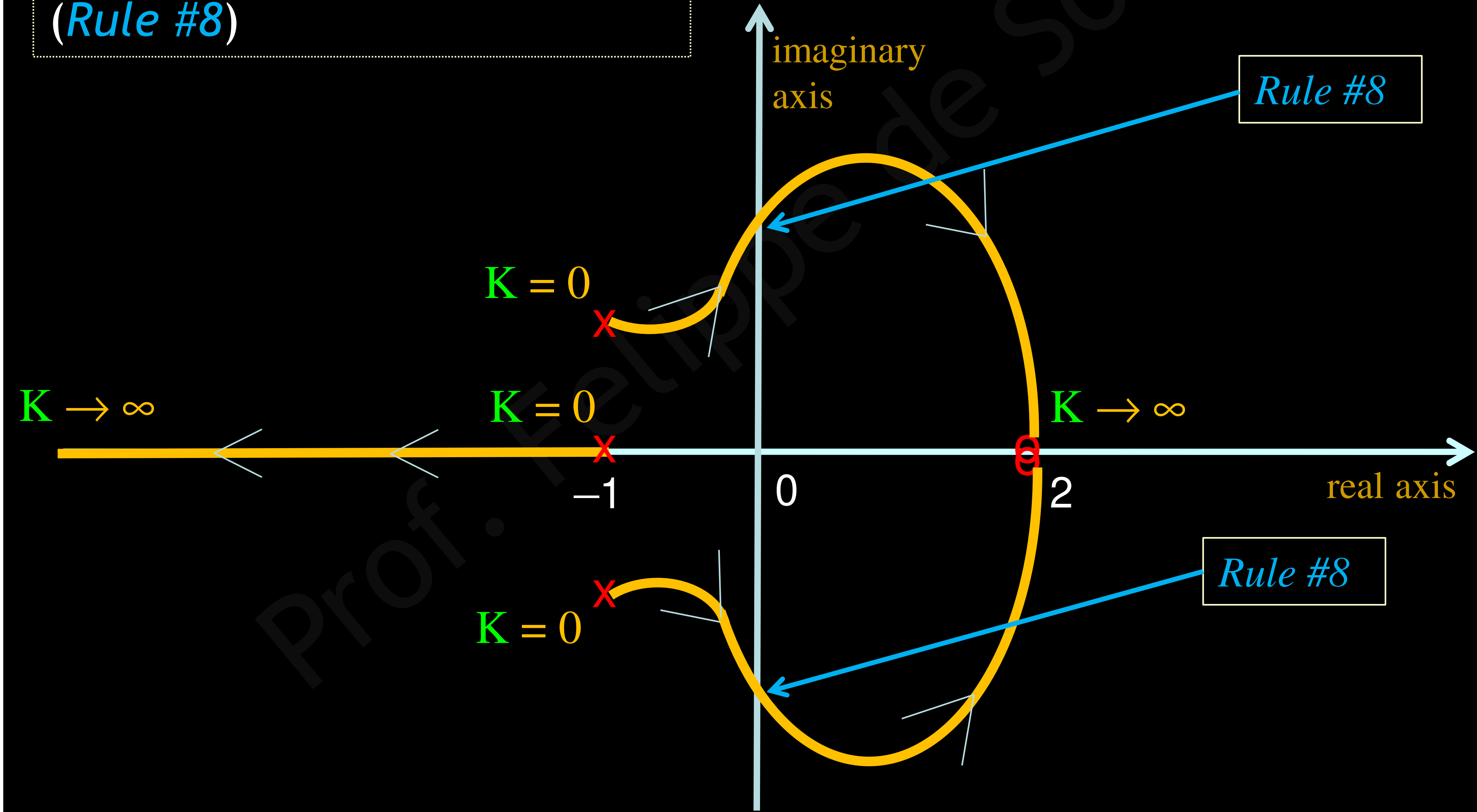
Example 14 (*continued*)

besides, a third *branch*, that start at the *real pole* at $s = -1$ goes to the left to lie on the *asymptote* at the ∞



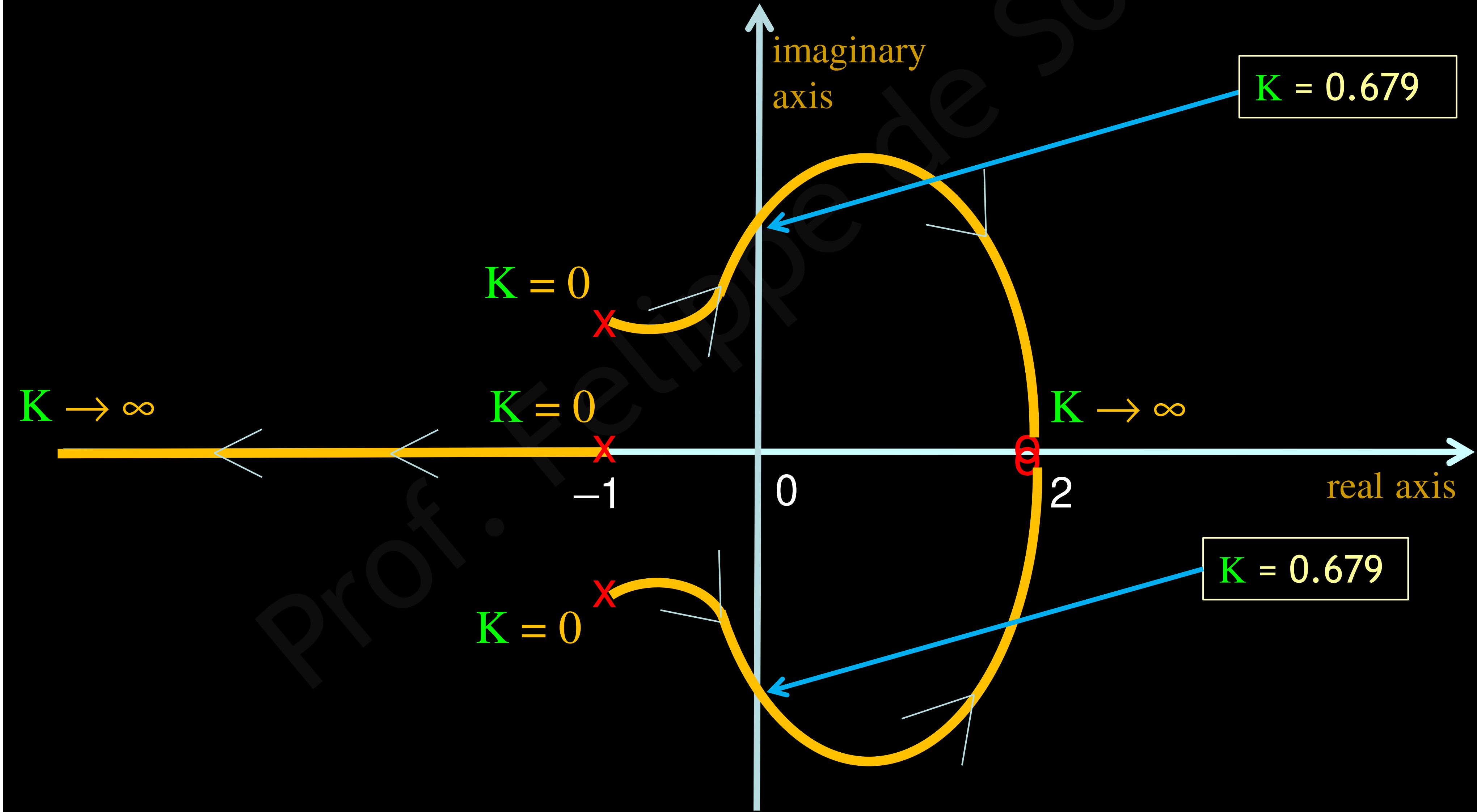
Example 14 (*continued*)

But it is still missing the *points* that the *Root Locus* intercepts the *imaginary axis* (*Rule #8*)



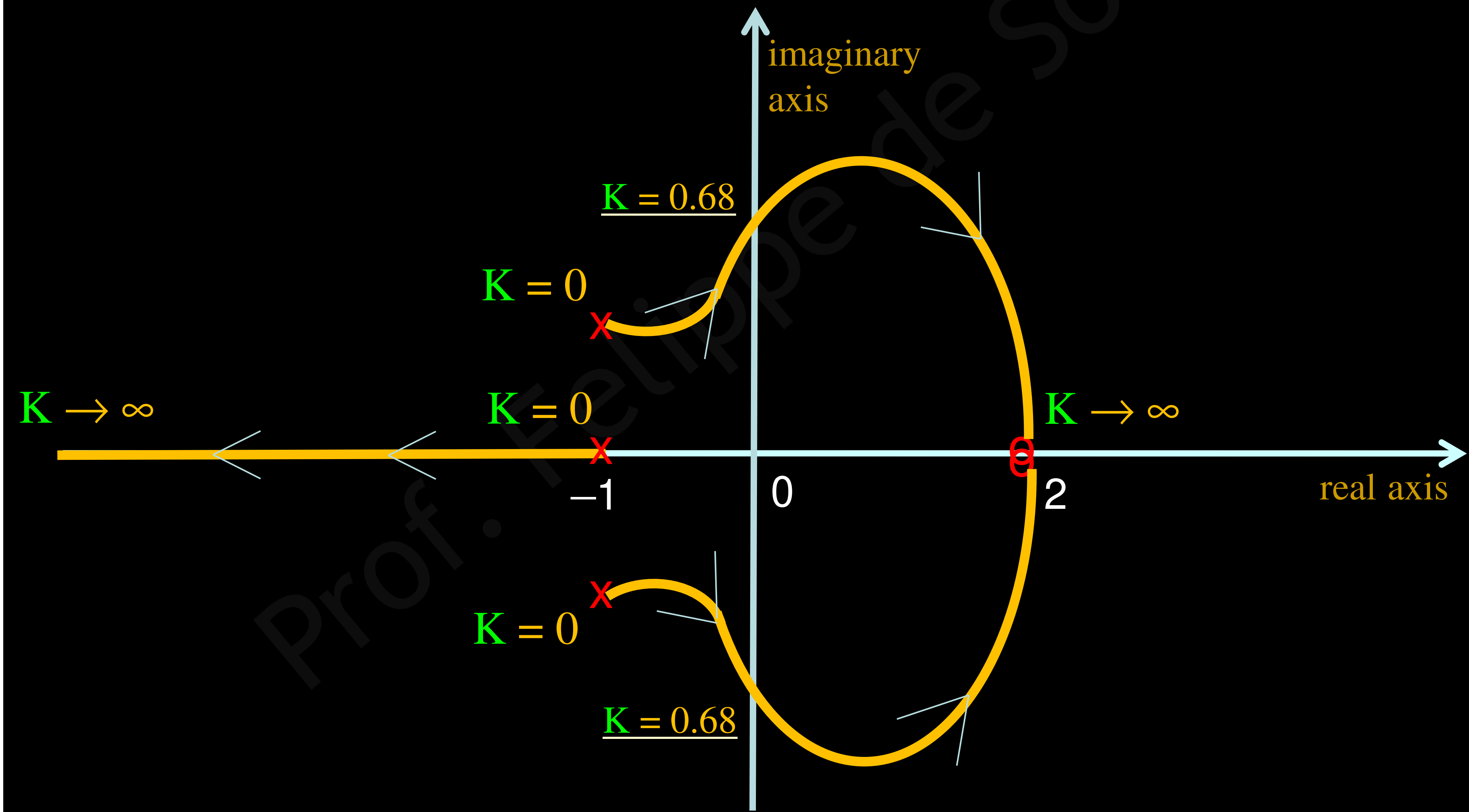
Example 14 (*continued*)

As seen in Example 11, the points from the imaginary axis where there are branches meeting (*Rule #8*) are: $s = \pm 1.132j$ (para $K \cong 0.68$)



Example 14 (*continued*)

Note that this *C.L. system* is *stable* only for $K < 0.68$



Example 15:

Sketching the "*Root Locus*" for

$$G(s)H(s) = \frac{K \cdot (s-2)^2}{(s^2+2s+6)\cdot(s-0{,}5)}$$

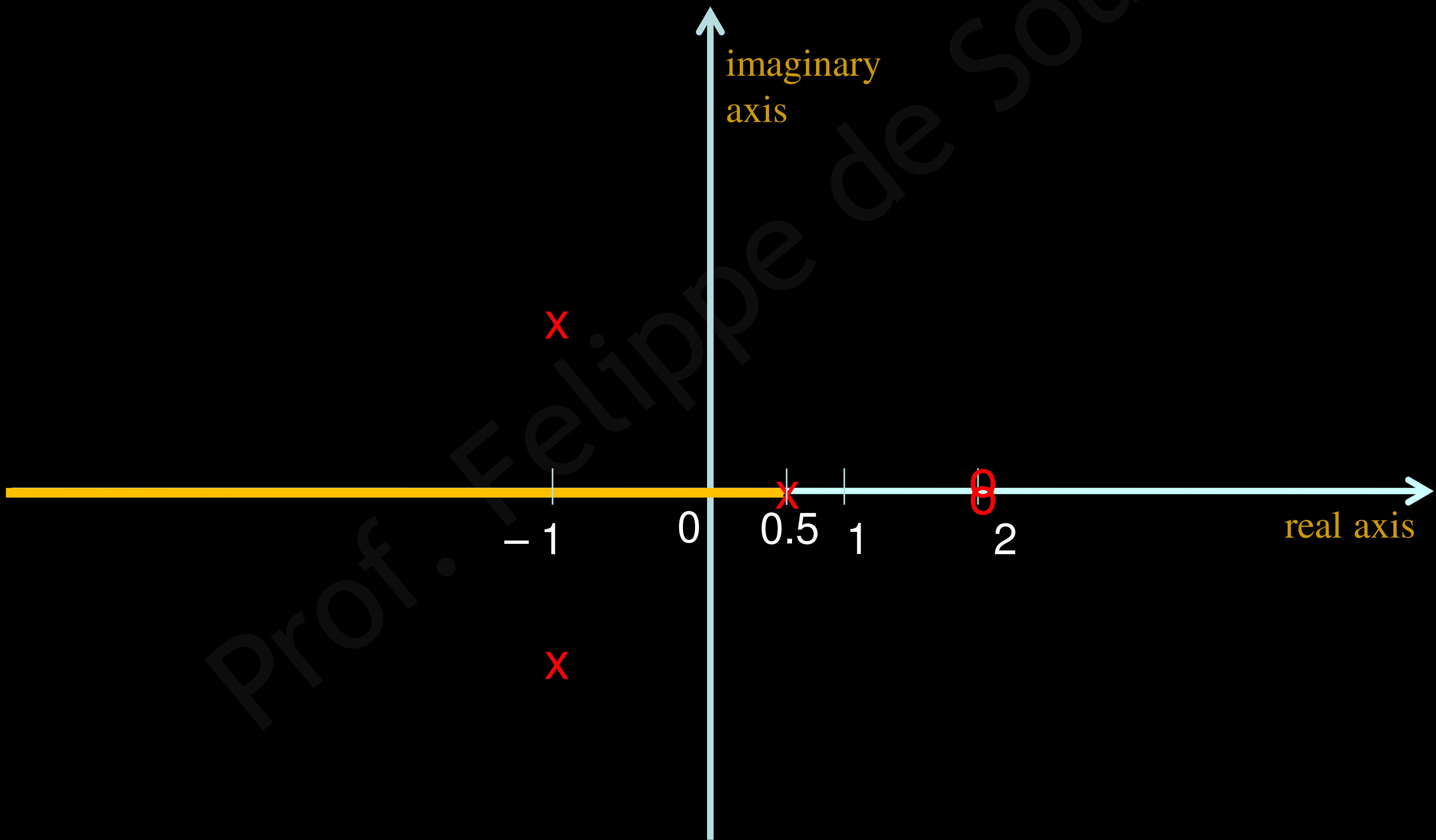
$n = 3$

$m = 2$

This "*Root Locus*" has 3 *branches* (*Rule #1*)

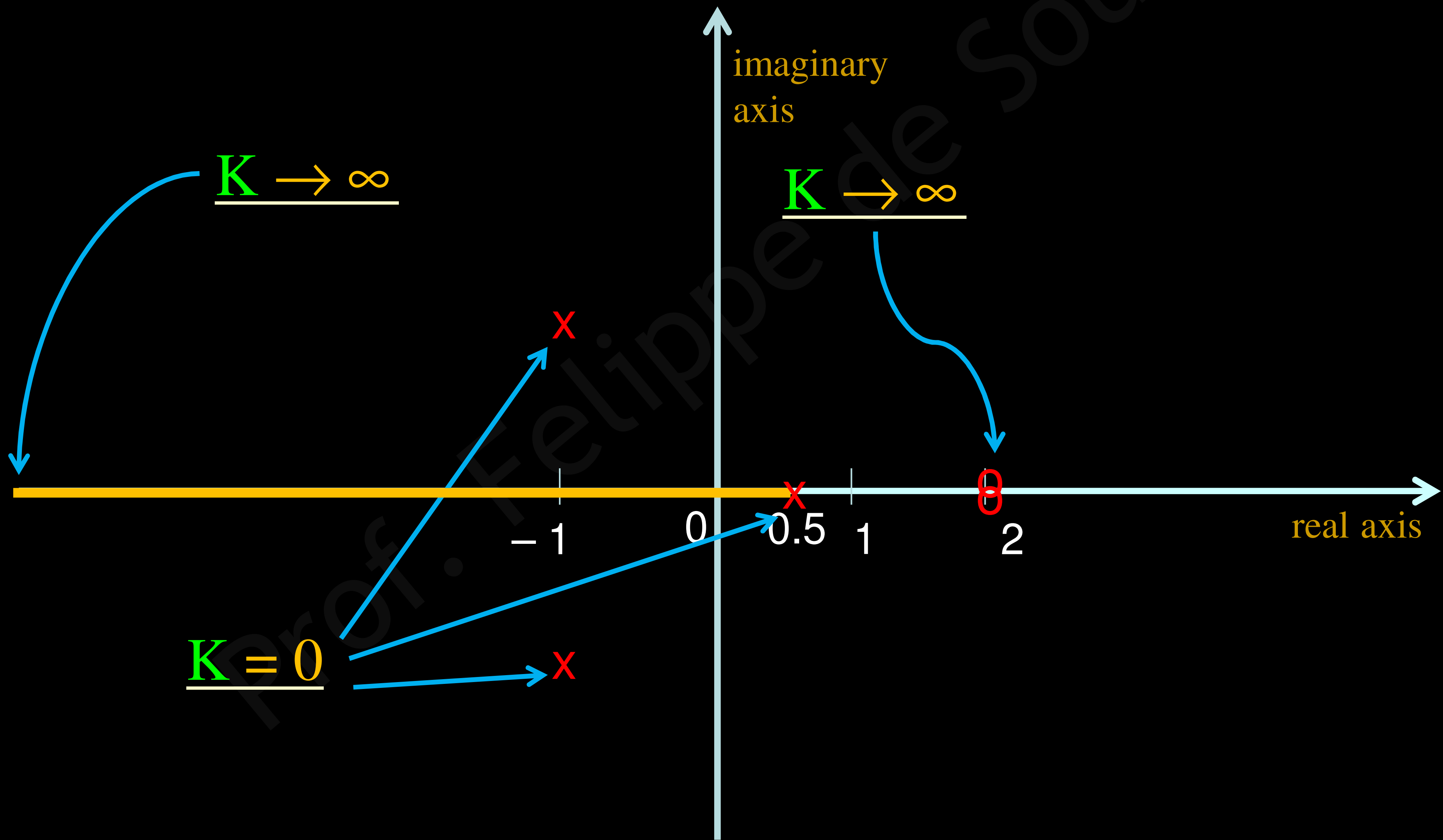
Example 15 (*continued*)

The *real axis* intervals (*Rule #2*)



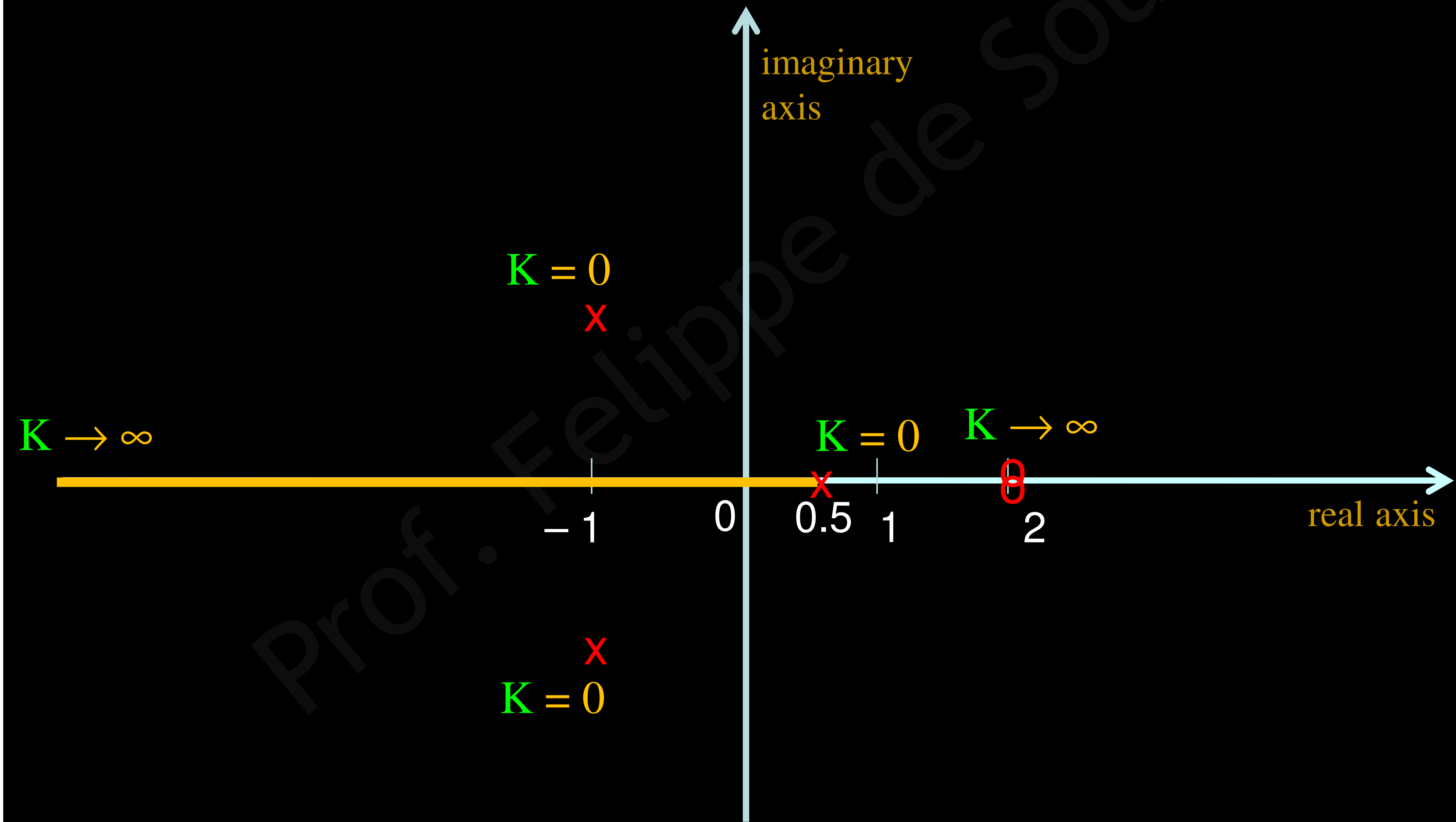
Example 15 (*continued*)

The 3 beginning *points* ($K = 0$) and ending *points* ($K \to \infty$) of this Root Locus (*Rule #3*) are shown below



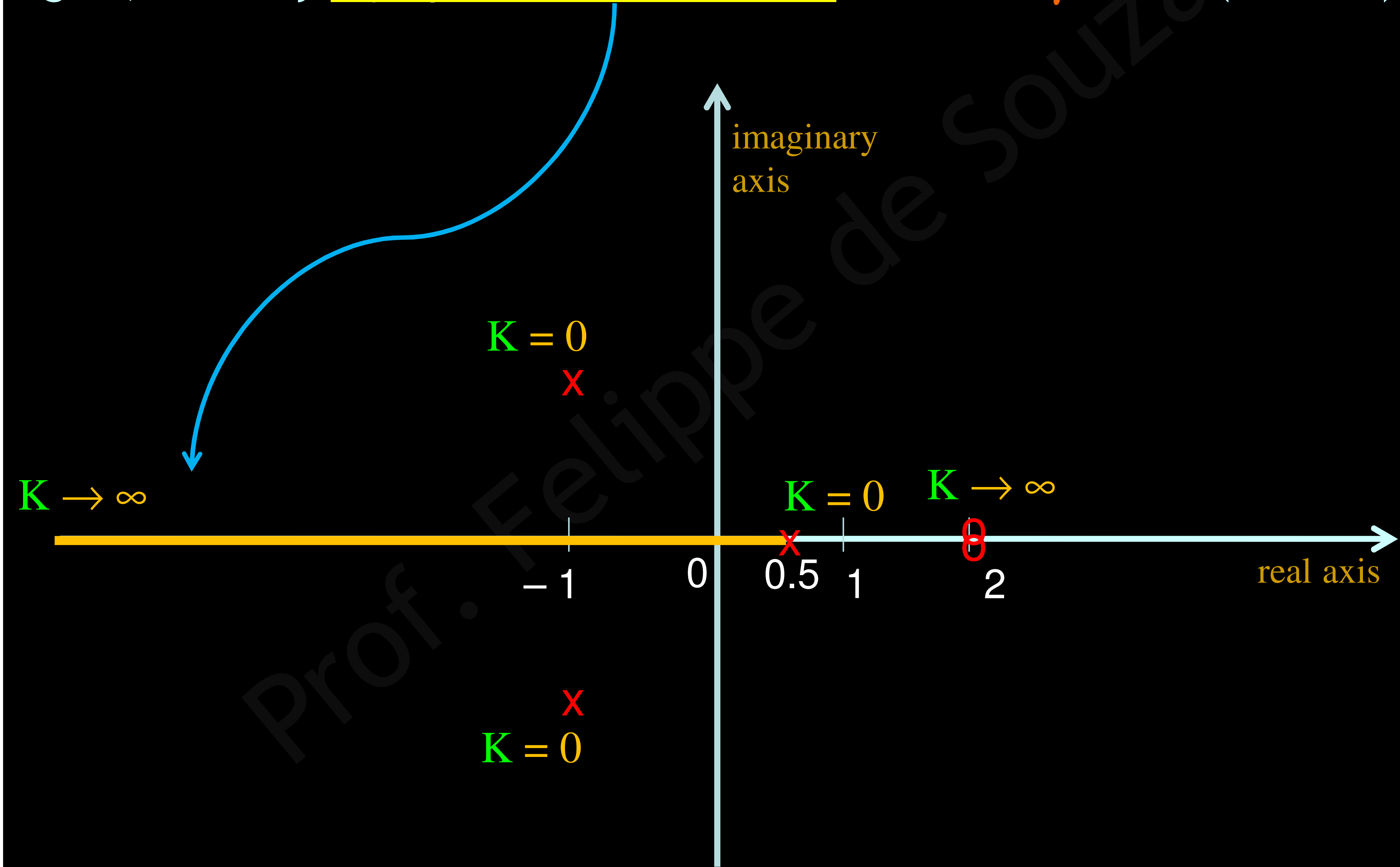
Example 15 (*continued*)

The 3 beginning *points* ($K = 0$) and ending *points* ($K \to \infty$) of this Root Locus (*Rule #3*) are shown below



Example 15 (*continued*)

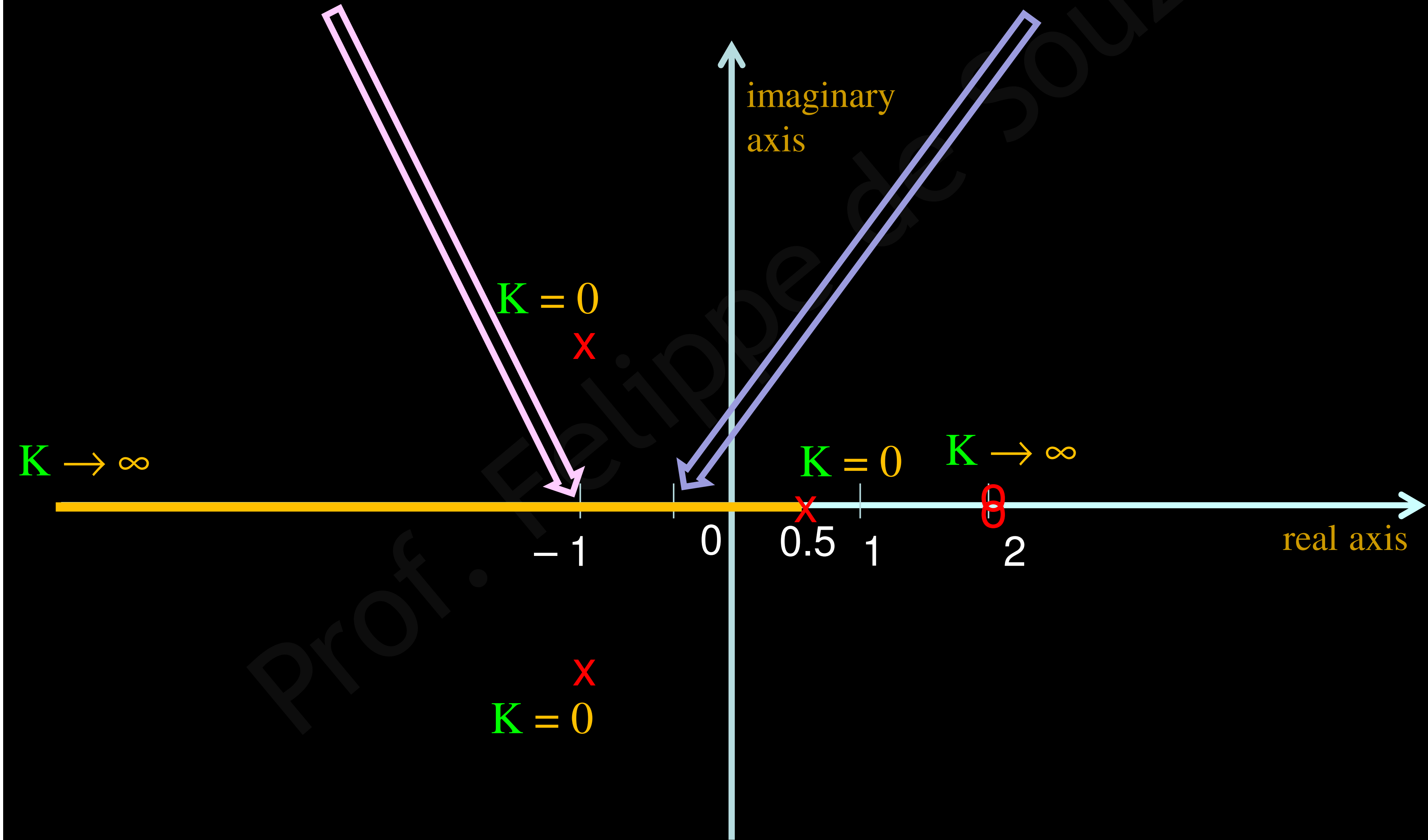
Again, the only *asymptote* at the *infinite* occurs at γ = 180º (*Rule #5*)



Example 15 (*continued*)

By *Rule #6*, this *Root Locus* has *branches* meeting at

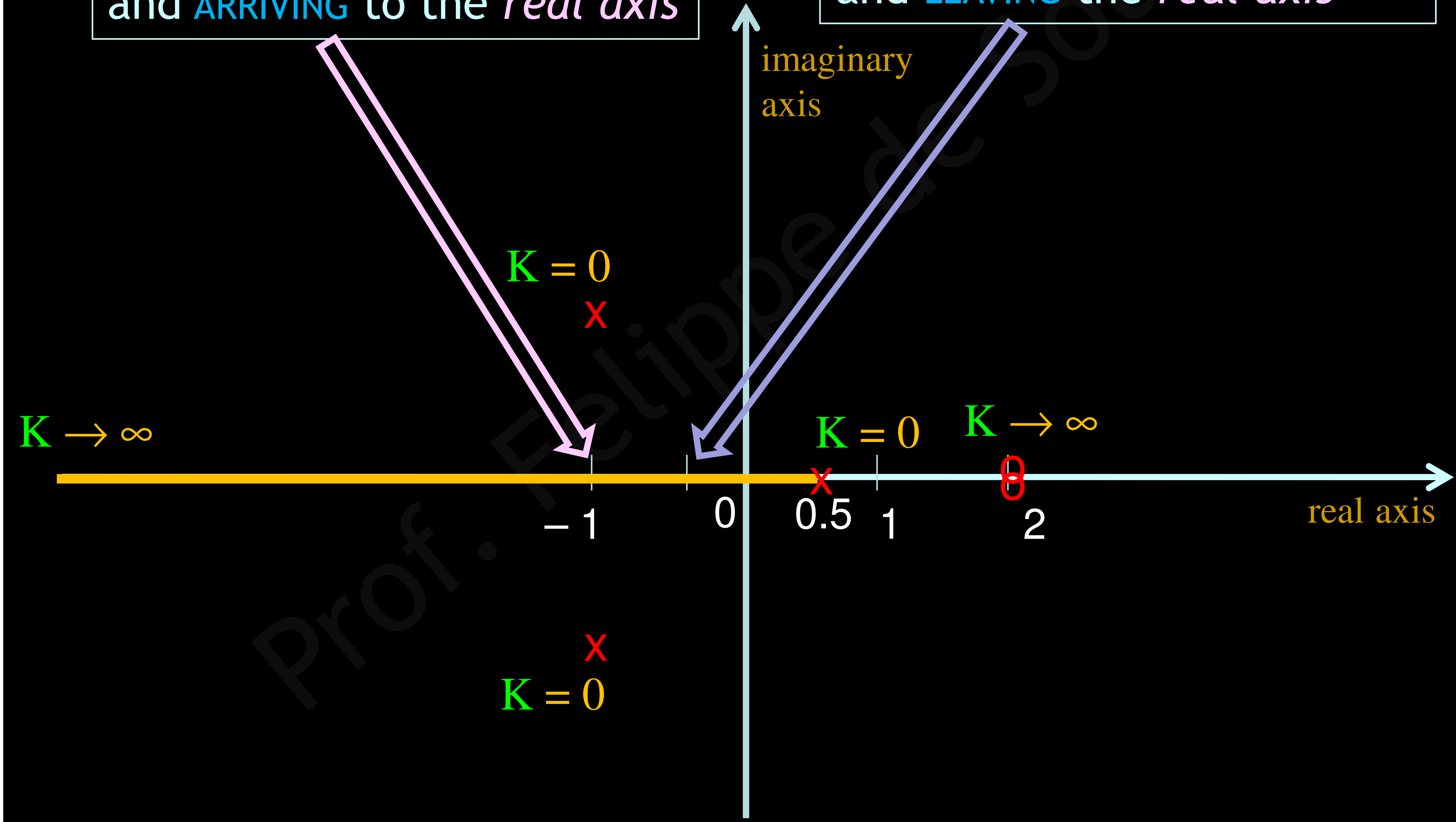
$s = -1$ (for $K = 0.833$) and $s = -0.531$ (for $K = 0.84$)



Example 15 (*continued*)

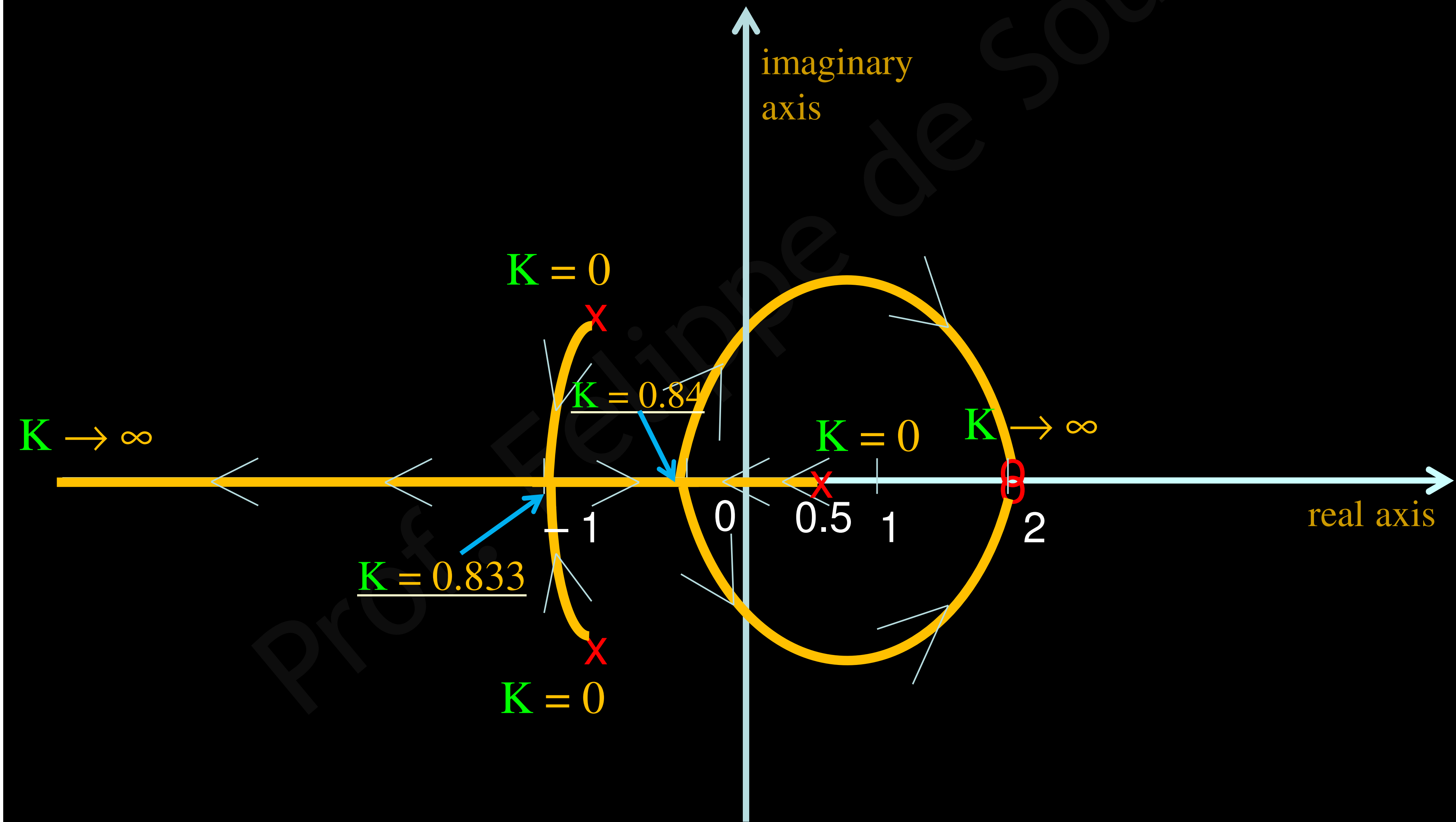
$s = -1$ (for $K = 0.833$)
branches that are meeting
and ARRIVING to the *real axis*

$s = -0.531$ (for $K = 0.84$)
branches that are meeting
and LEAVING the *real axis*



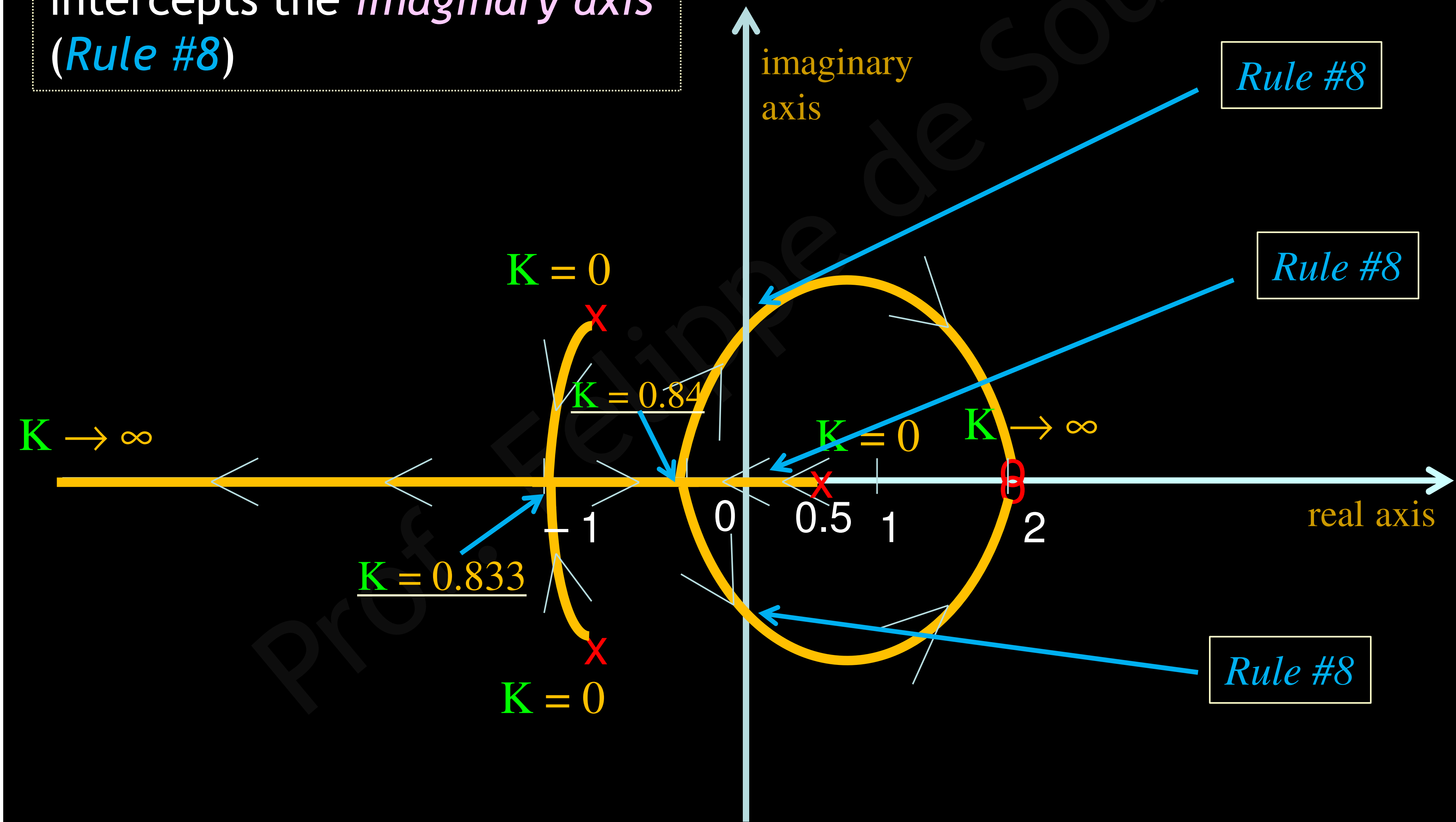
Example 15 (*continued*)

So, the *Root Locus* completed will have the following aspect



Example 15 (*continued*)

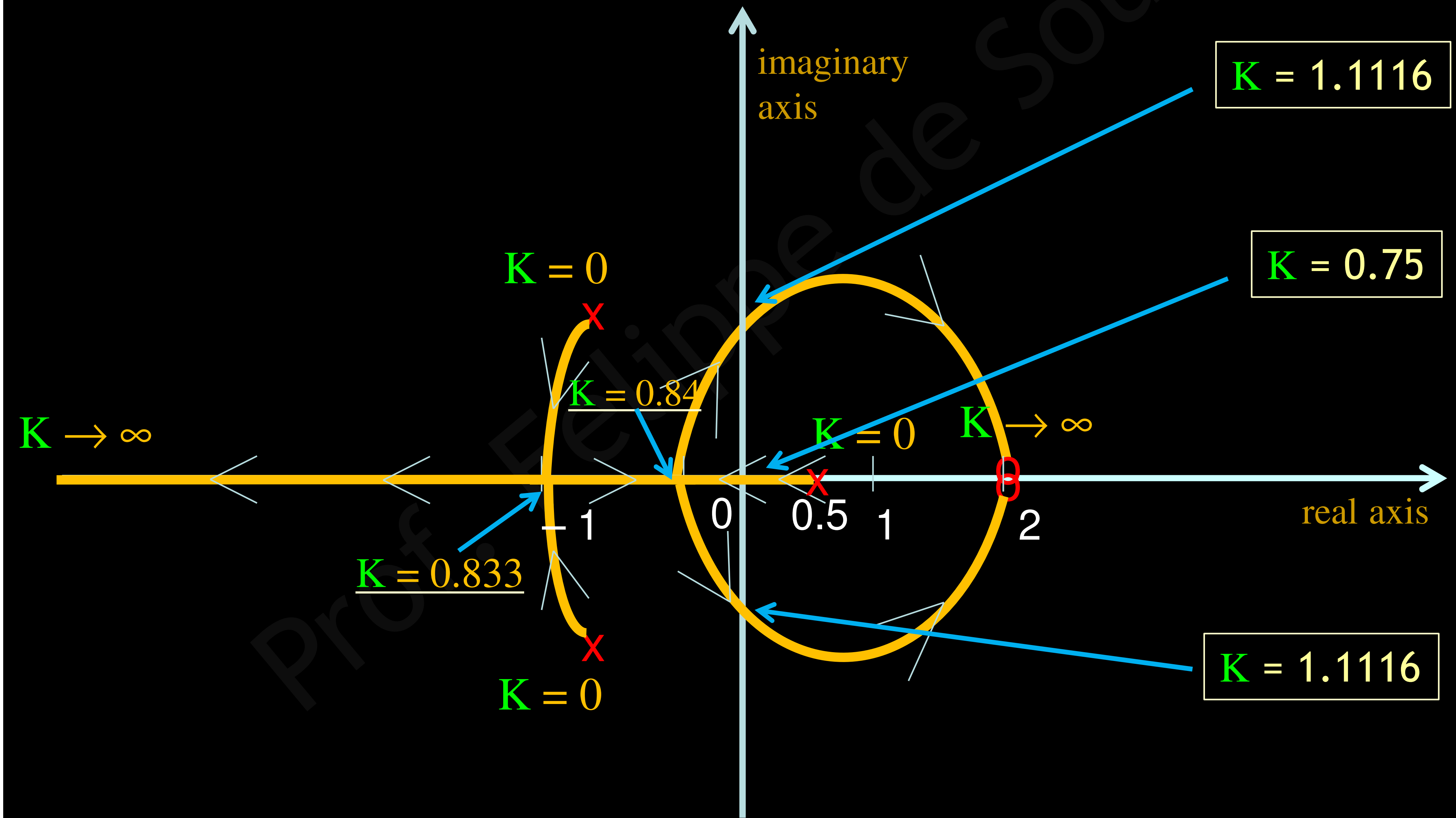
But it is still missing the *points* that the *Root Locus* intercepts the *imaginary axis* (*Rule #8*)



Example 15 (*continued*)

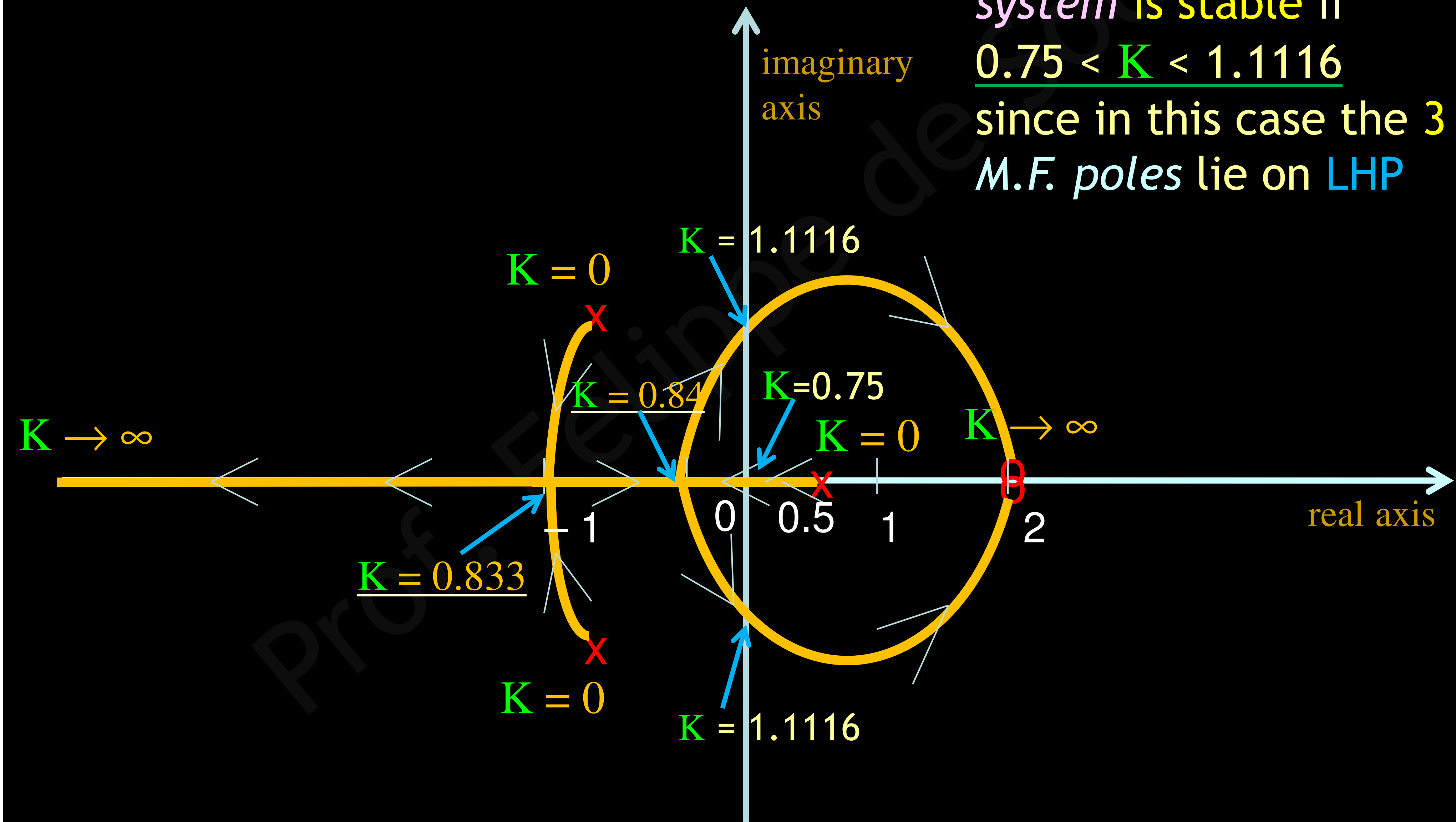
And the values of K that makes *pivot column elements* to vanish are

K = 0.75 and K = 1.1116



Example 15 (*continued*)

and the *Root Locus* is then completed



Observe that the *C.L. system* is stable if $0.75 < K < 1.1116$ since in this case the 3 *M.F. poles* lie on LHP

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Obrigado!

Thank you!

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