Control Systems

11

"Root Locus"

part I

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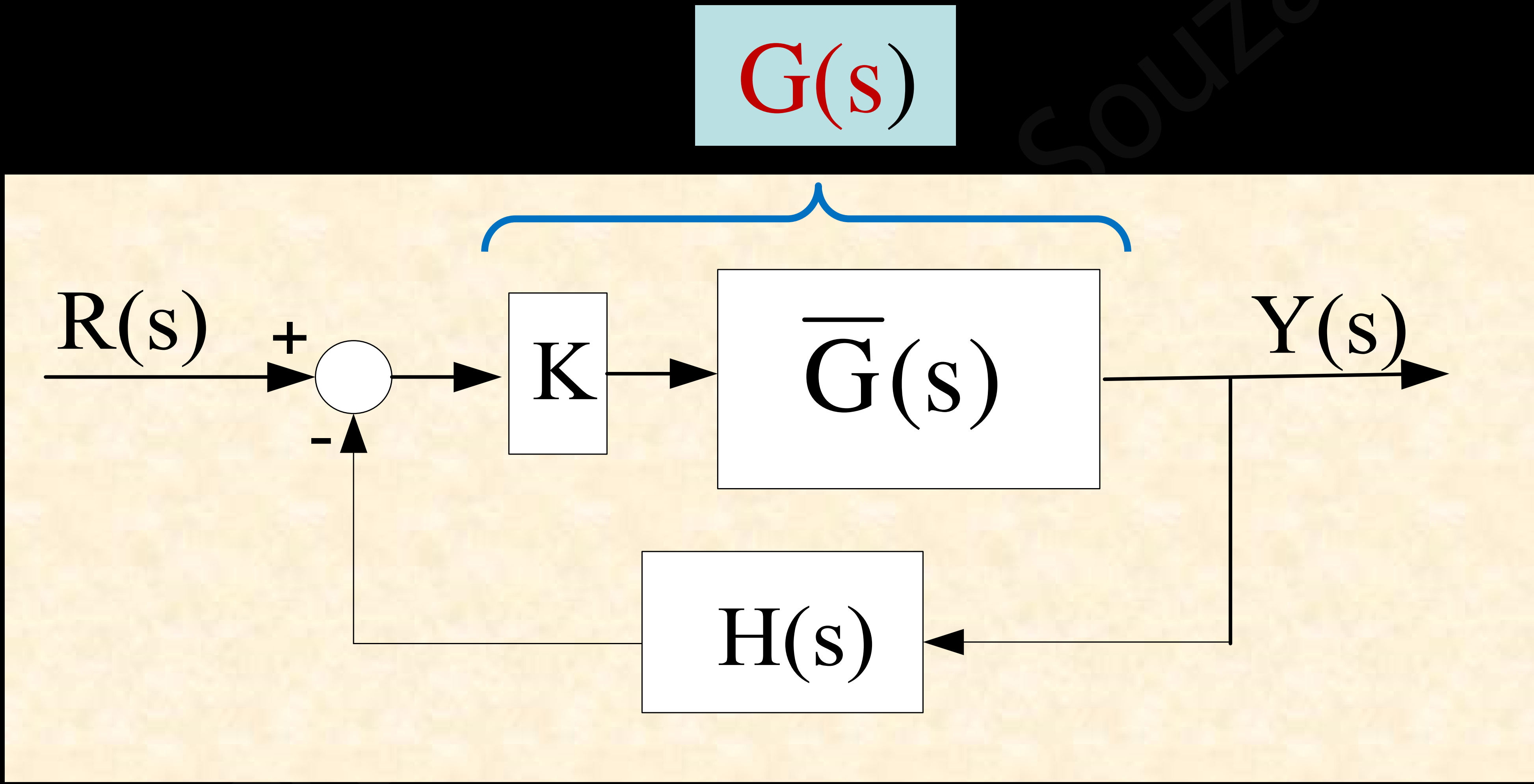
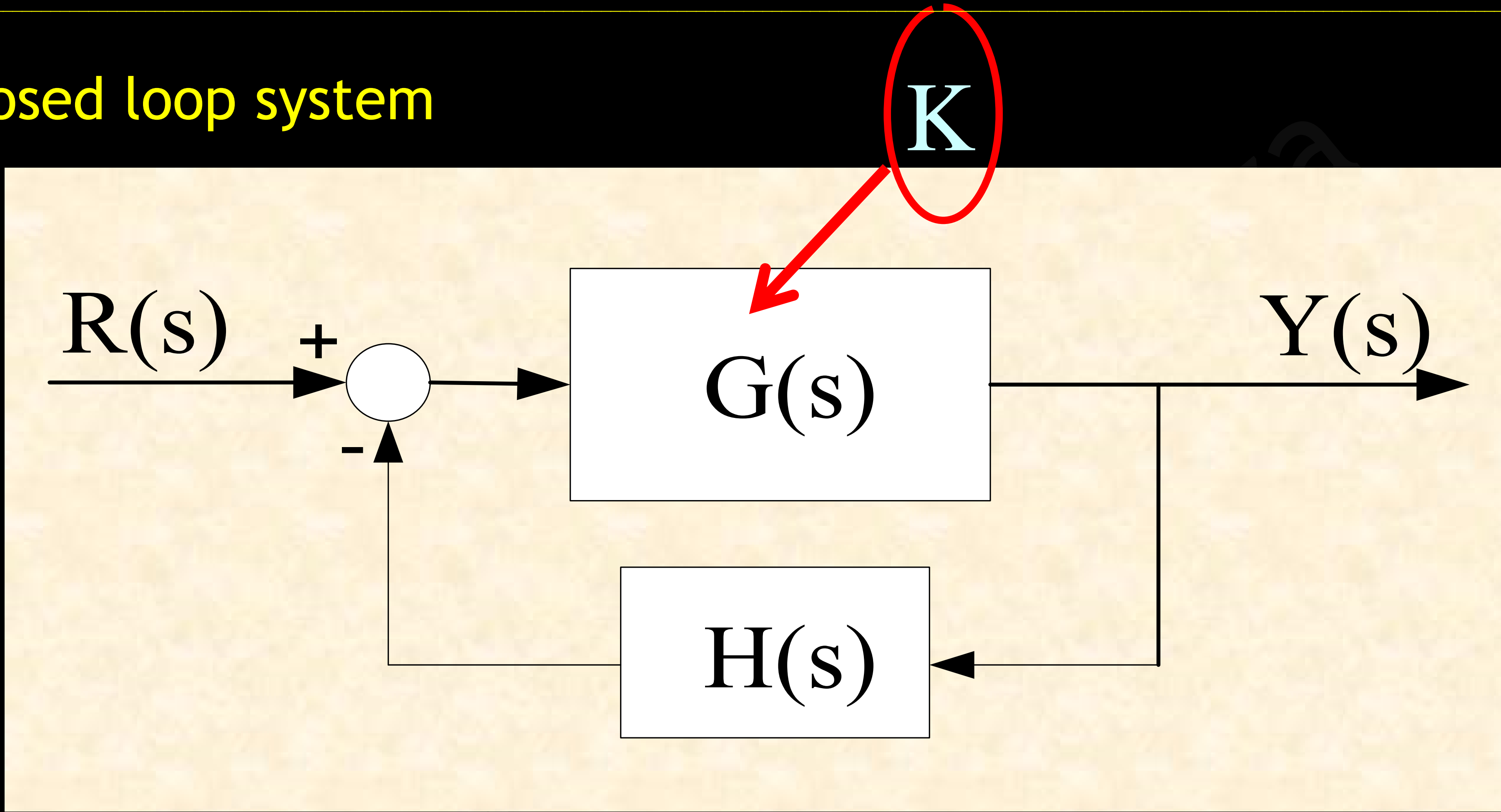
Closed loop system



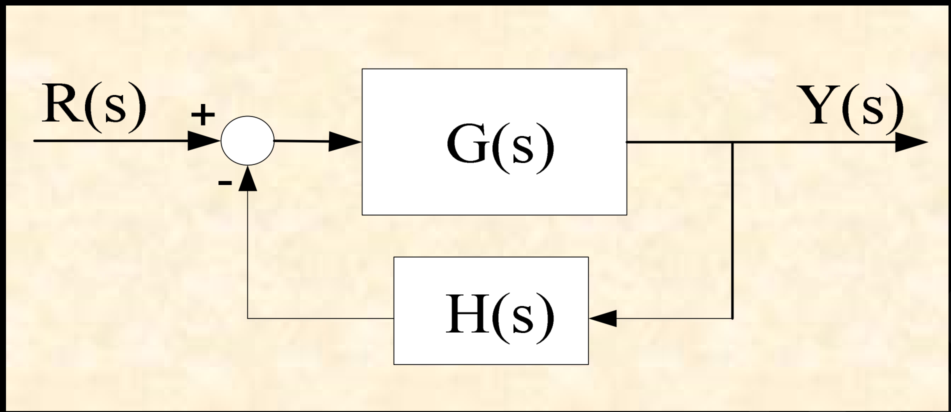
Closed loop system

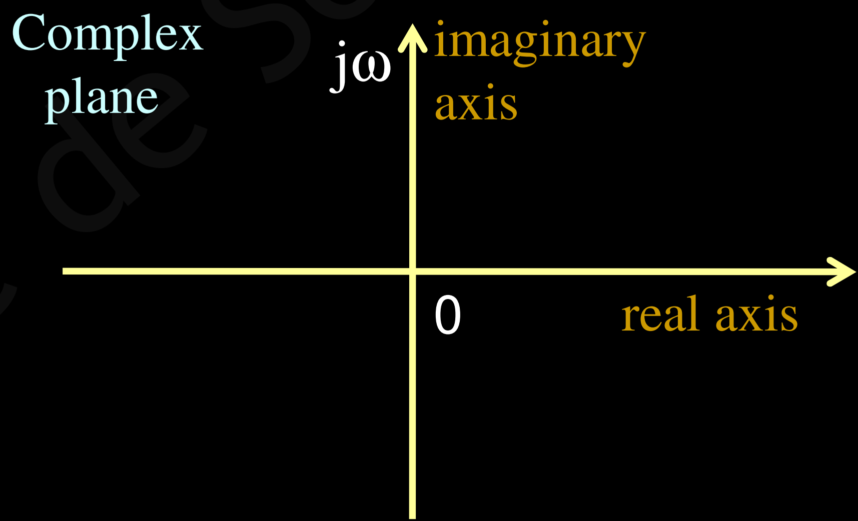


The "*Root Locus*" the locus of the *poles* of the *closed loop* system, when we vary the value of K

We shall assume here $K > 0$, but it is also possible to construct the "*Root Locus*" for $K < 0$ or even for $-\infty < K < \infty$.

"*Root Locus*" is actually the locus of the *poles*, that is, the locus of the *roots of the characteristic equation*, of the closed loop *system*





Thus, the "*Root Locus*" is drawn in the *complex plane*

It is easy to observe that the "*Root Locus*" is SIMETRIC with respect to the *real axis*

That is, the *upper part* is a reflex of the *lower part*

As we know very well by now, the closed loop "*transfer function*" (CLTF) of the system is given by

$$\mathrm{F.T.} = \frac{G(s)}{1 + G(s)H(s)}$$

and the closed loop *poles* of this *system* are the *roots* of the characteristic equation of the CLTF

It is easy to show that these *roots* of the *characteristic equation* of the CLTF are the same *roots* of

$$1 + G(s) \cdot H(s) = 0$$

That is, we calculate the expression

$$1 + G(s) \cdot H(s) = 0$$

and then calculate the *roots of the* numerator.

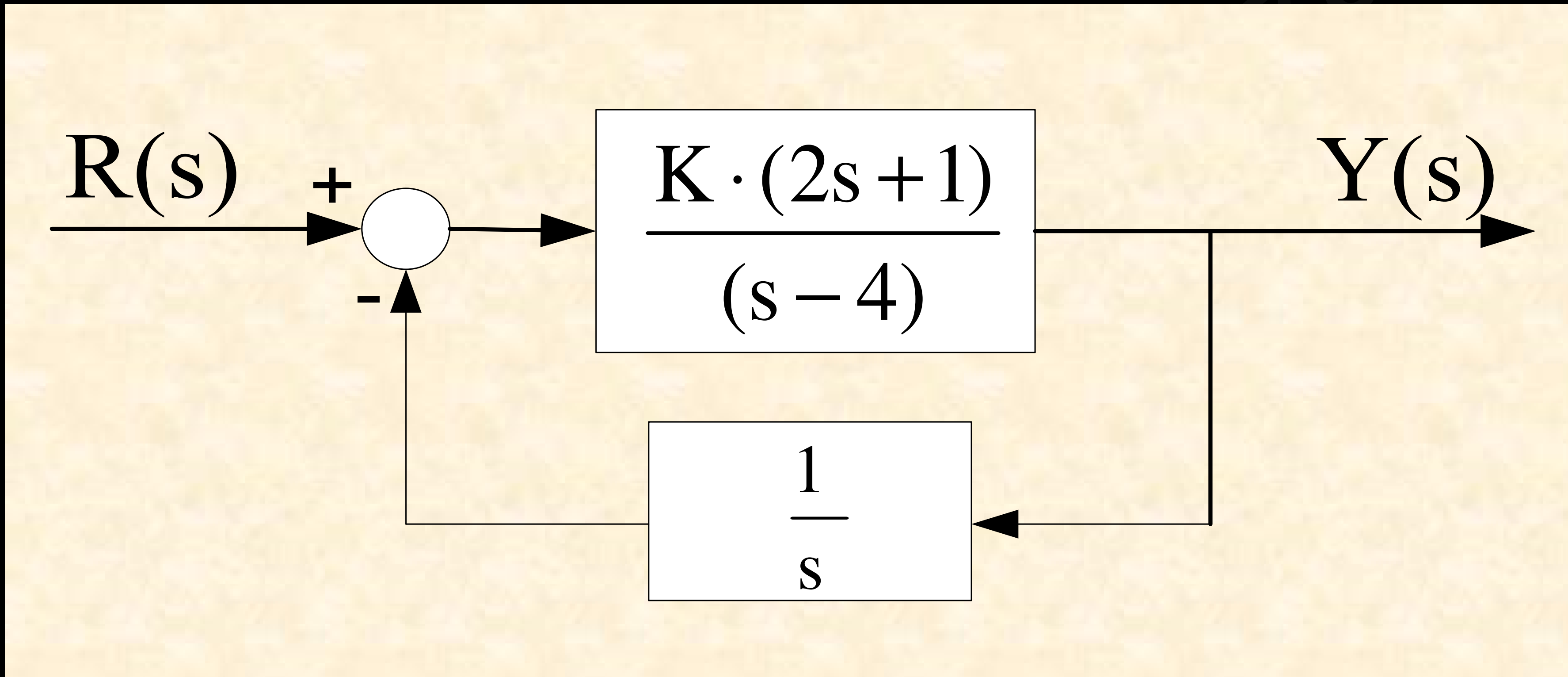
Those will be the *roots of the characteristic equation* of the CLTF

without having to calculate the CLTF

Actually, the *characteristic equation* of this CLTF is precisely the numerator of

$$1 + G(s) \cdot H(s)$$

Example 1: Let us calculate the *characteristic equation* of the C.L. *system* below



Observe that:

$$1 + G(s)H(s) \;=\; 1 \;+\; \frac{K\cdot(2s+1)}{(s-4)s}$$

Example 1 (*continued*)

hence,

$$1 + G(s)H(s) = \frac{s^2 + (2K-4)s + K}{(s-4)s}$$

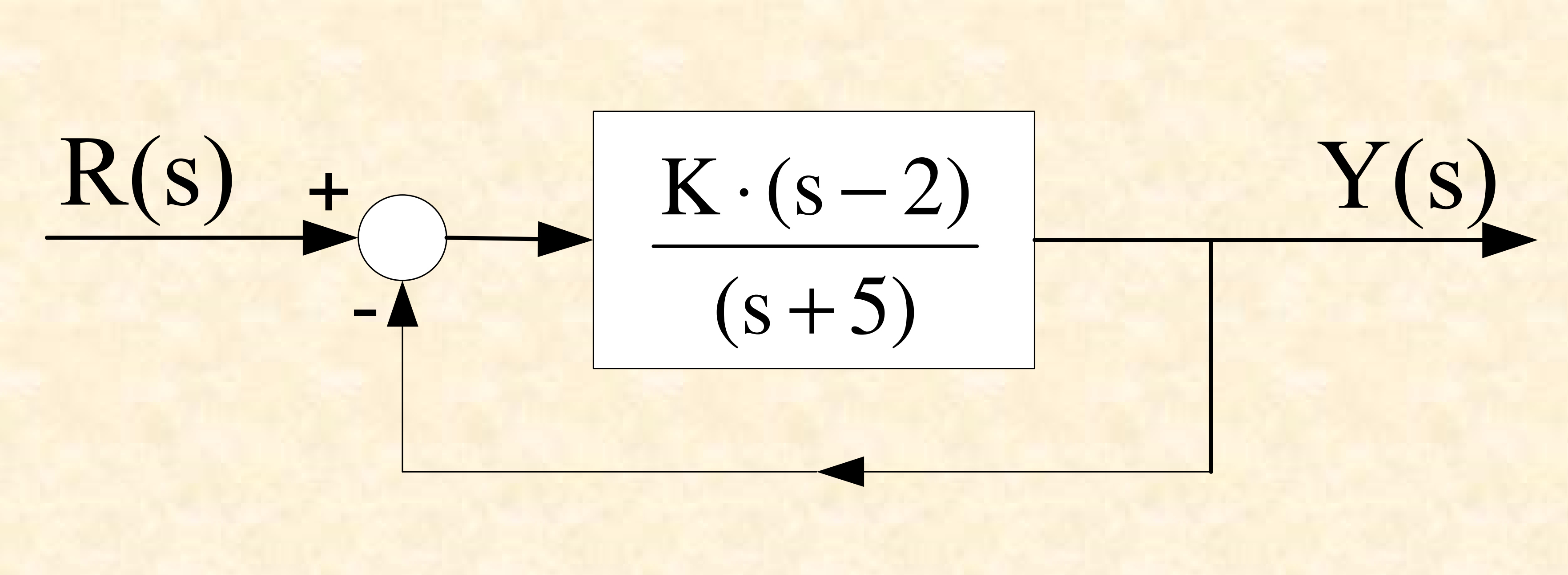
and therefore, *the characteristic equation* of the CLTF is given by:

$$s^2 + (2K-4)s + K = 0$$

That could also be obtained (*with a little more calculation*) through the denominator of the CLTF which is given by:

$$FTMF = \frac{K(2s+1)s}{s^2 + (2K-4)s + K}$$

Example 2: Let us construct the *Root Locus* of the C.L. *system*

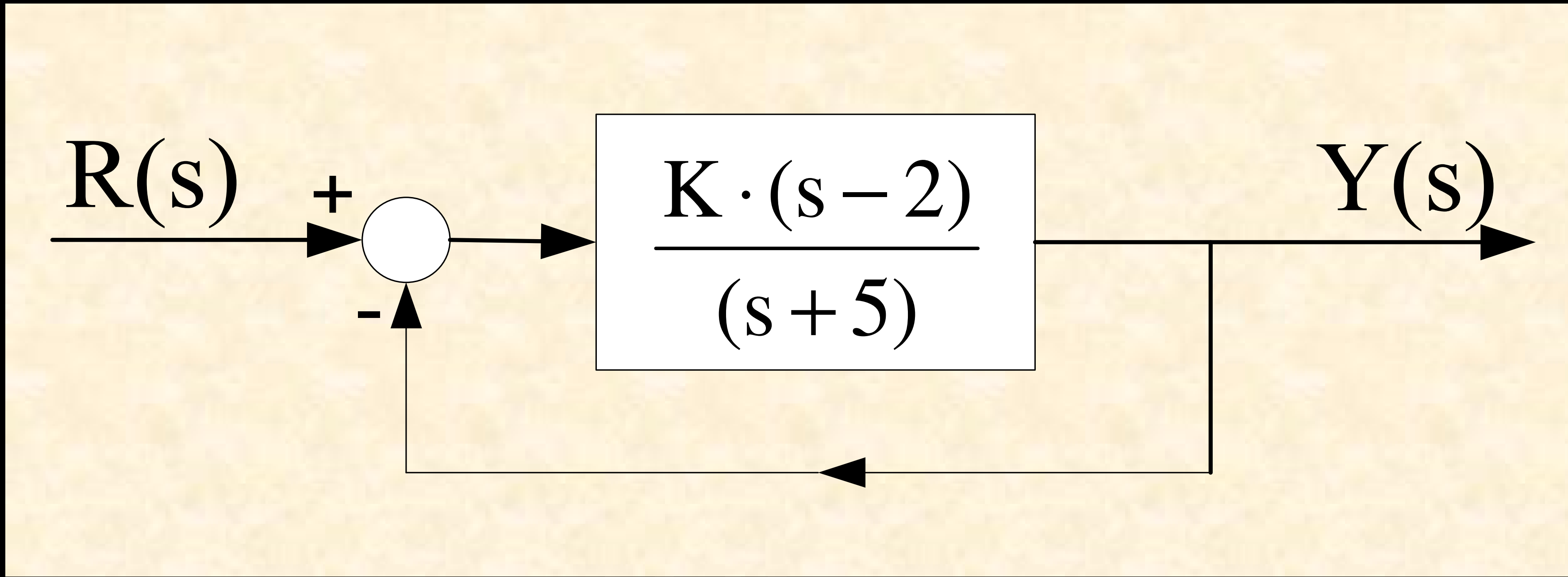


Observe that:

$$1+G(s)H(s) = 1 + \frac{K\cdot(s-2)}{(s+5)}$$

$$= \frac{(K+1)s + (5-2K)}{(s+5)}$$

Example 2 (*continued*)



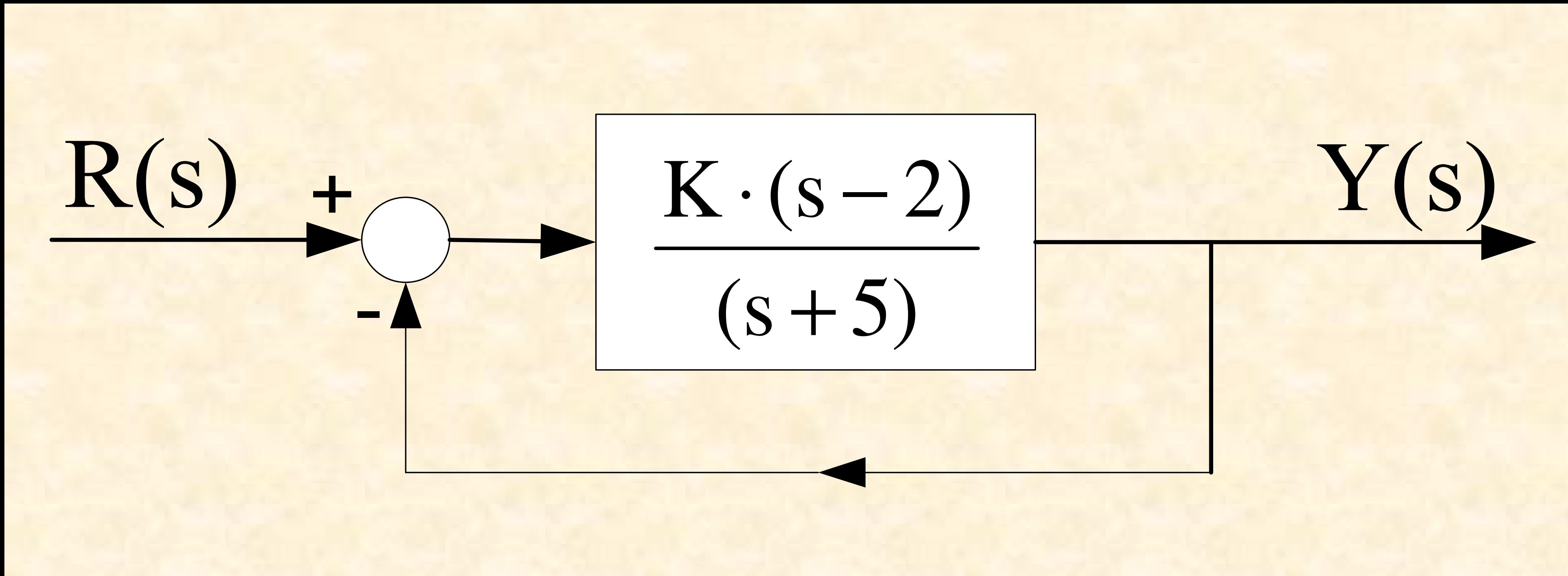
thus, the *characteristic equation* of the C.L. system is:

$$(K+1)\,s \;+\; (5-2K) \;=\; 0$$

and the only C.L. pole is:

$$s \;=\; \frac{(2K-5)}{(K+1)}$$

Example 2 (*continued*)



Therefore, the Root Locus of this *system* is the *locus* of s

$$s = \frac{(2K-5)}{(K+1)}$$

in the *complex plane*, when K varies, for $K > 0$.

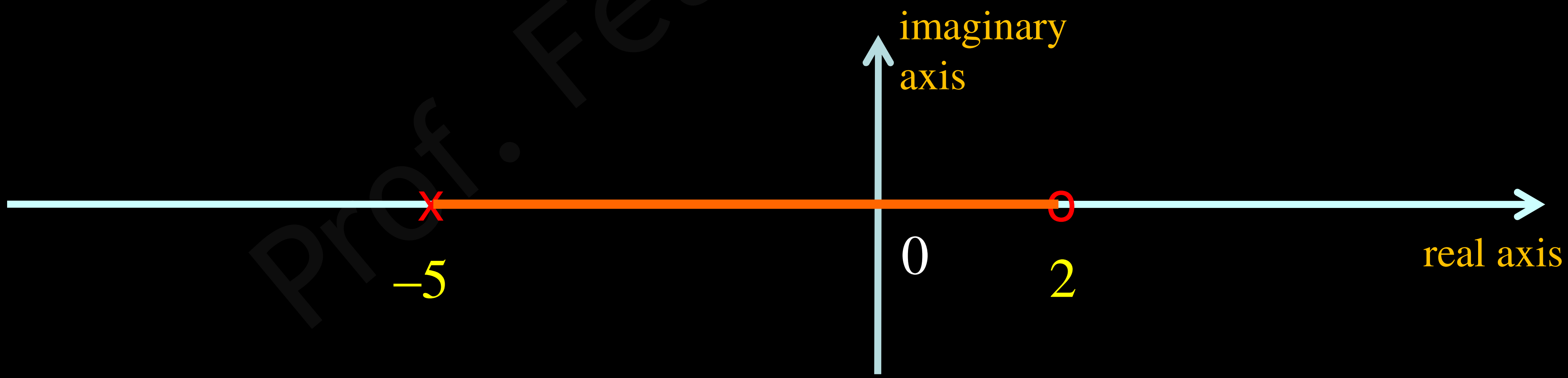
Example 2 (*continued*)

Observe that:

if $K = 0$ ⟹ $s = -5$

if $K \to \infty$ ⟹ $s \to +2$

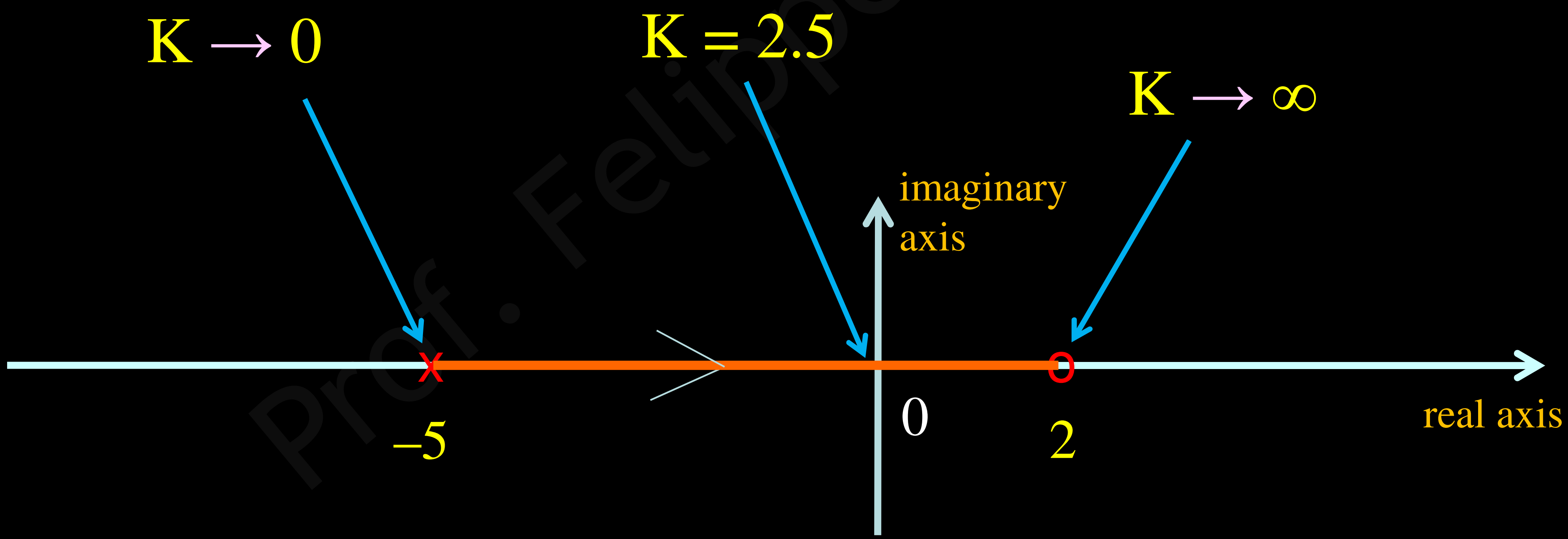
Hence, it is easy to observe that the Root Locus of this *system* is the *line segment* in the real axis between -5 and $+2$, that is $[-5, +2]$.



Example 2 (*continued*)

Actually, the *line segment* that goes from -5 to 2, when K goes from 0 to ∞.

and note that $K = 2.5$ when $s = 0$.



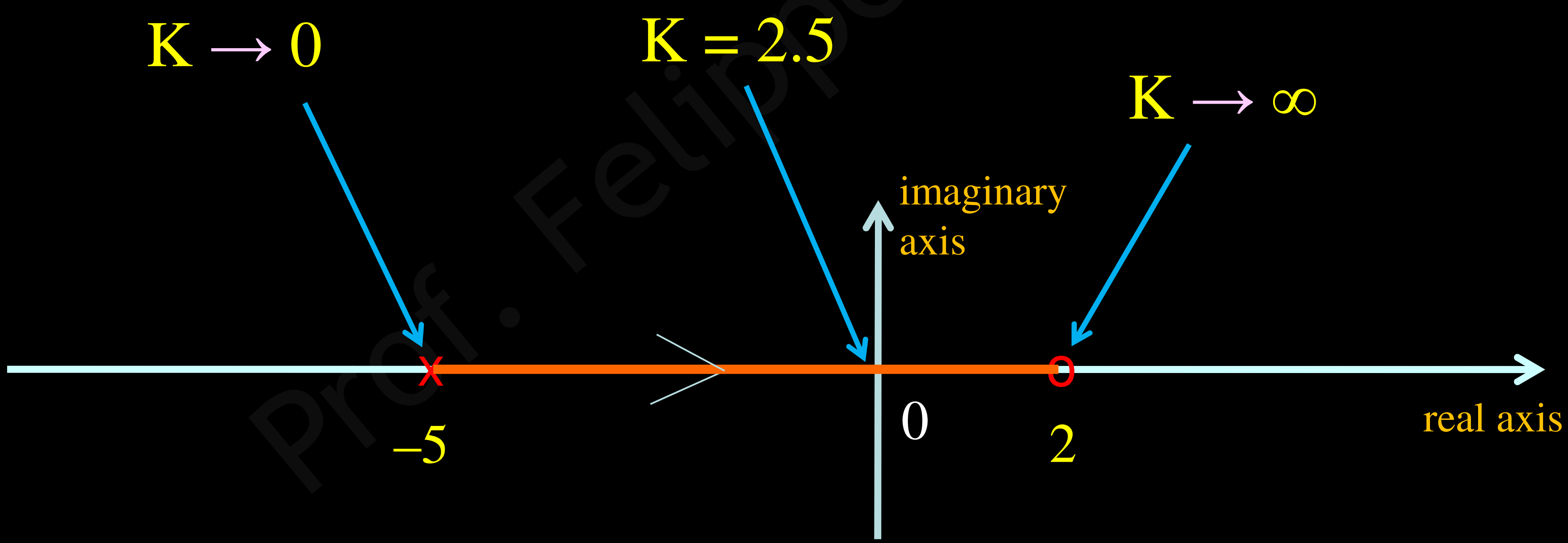
Example 2 (*continued*)

Since this a 1st order closed loop *system*,

the sole closed loop pole is real,

and therefore,

this Root Locus is entirely located in the *real axis*.



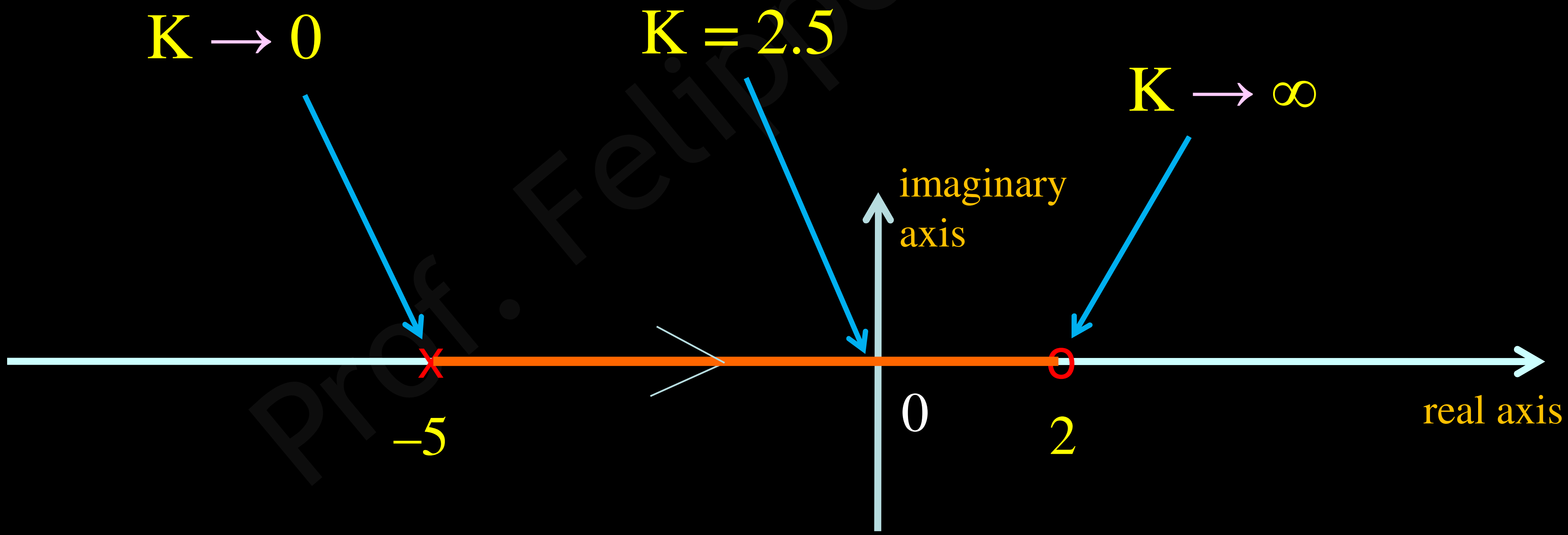
Example 2 (*continued*)

The Root Locus allows us to see that

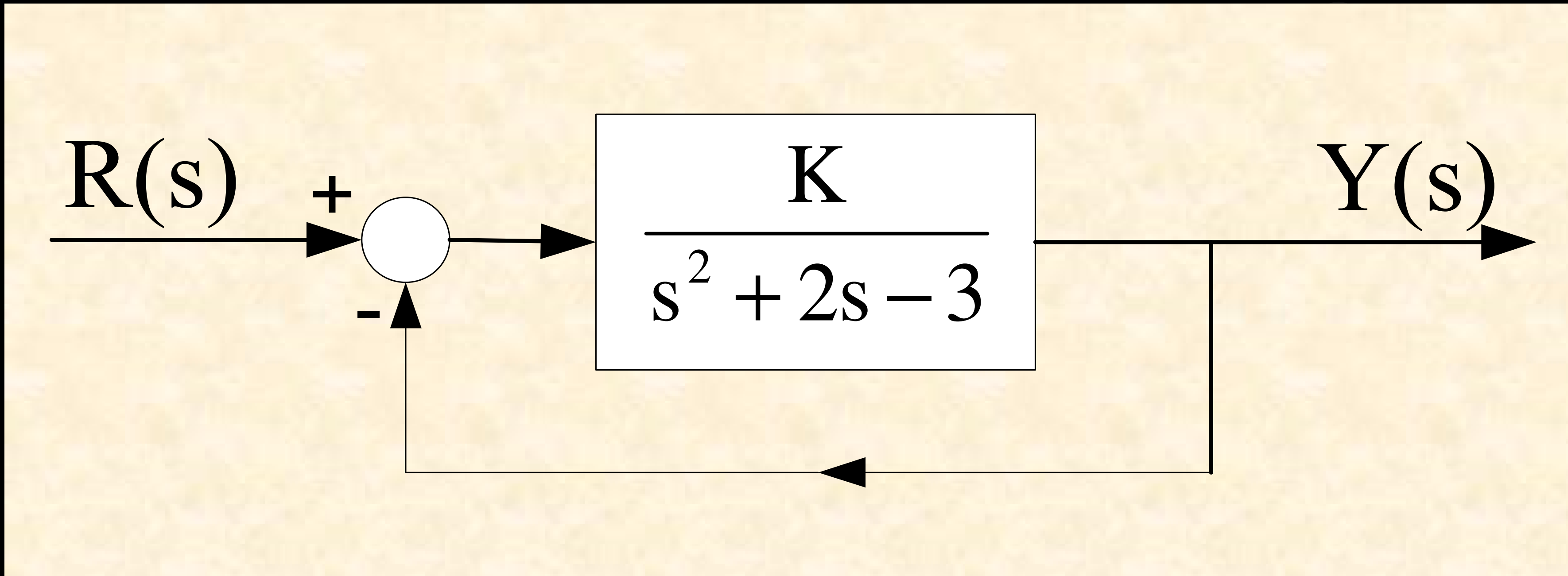
for $K < 2.5 \quad \Rightarrow \quad$ the *closed loop system* is stable,

since in this case the only closed loop pole will lie in the LHP, whereas

for $K \geq 2.5 \quad \Rightarrow \quad$ the *closed loop system* is not stable.



Example 3: Let us construct the *Root Locus* of the C.L. *system*

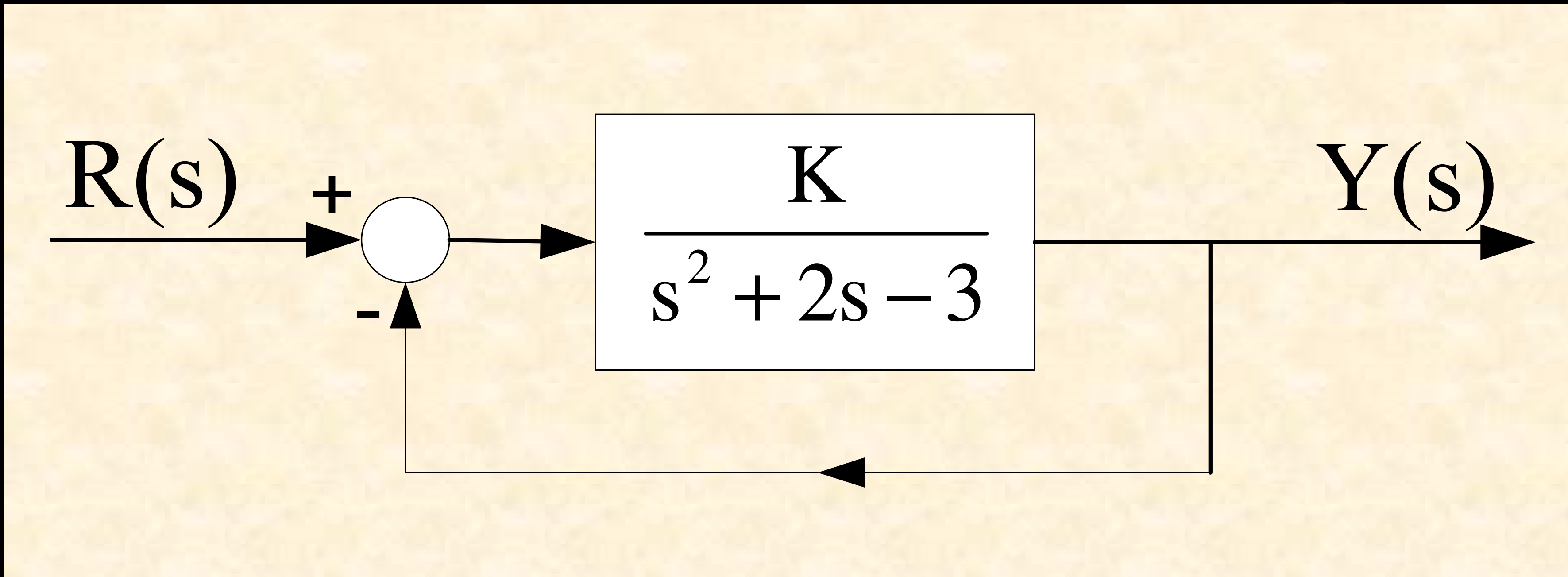


Observe that:

$$1 + G(s)H(s) = 1 + \frac{K}{(s-1)(s+3)}$$

$$= \frac{s^2 + 2s + (K-3)}{s^2 + 2s - 3}$$

Example 3 (*continued*)



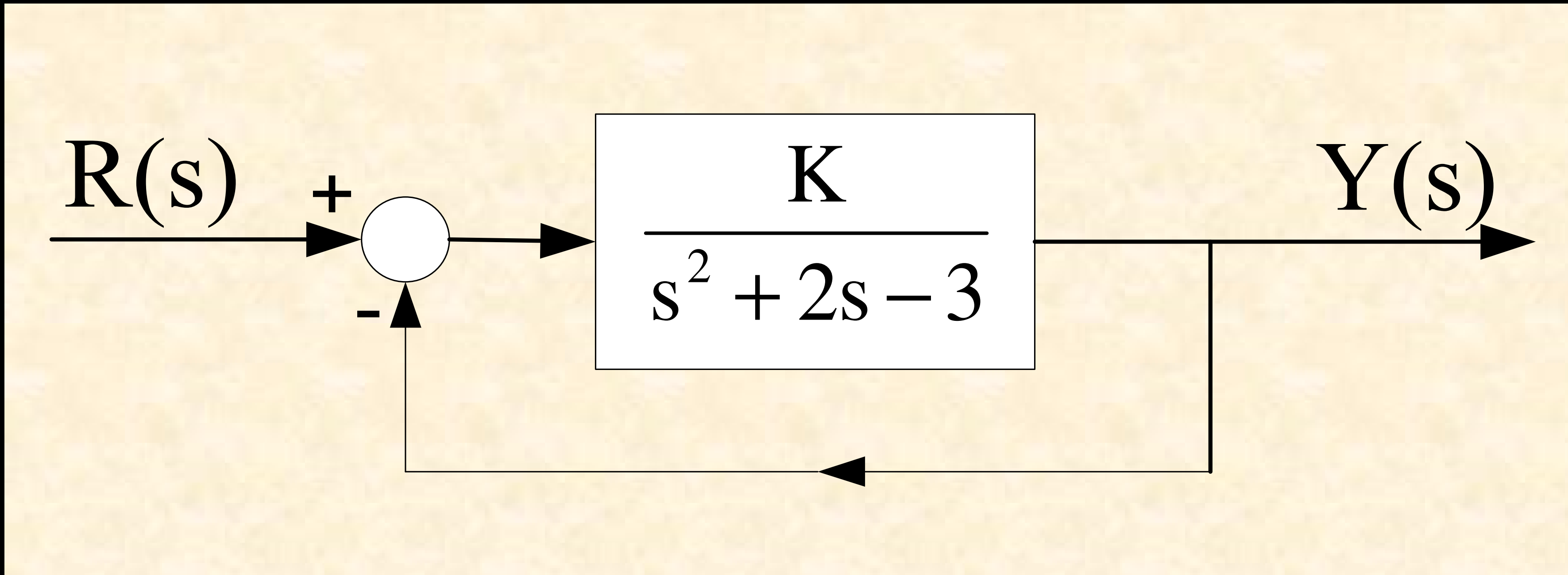
thus, the *characteristic equation* of the C.L. system is:

$$s^2 + 2s + (K - 3) = 0$$

and the 2 poles are:

$$s = -1 \pm \sqrt{4 - K}$$

Example 3 (*continued*)



So, the Root Locus of this *system* is the *locus* of s

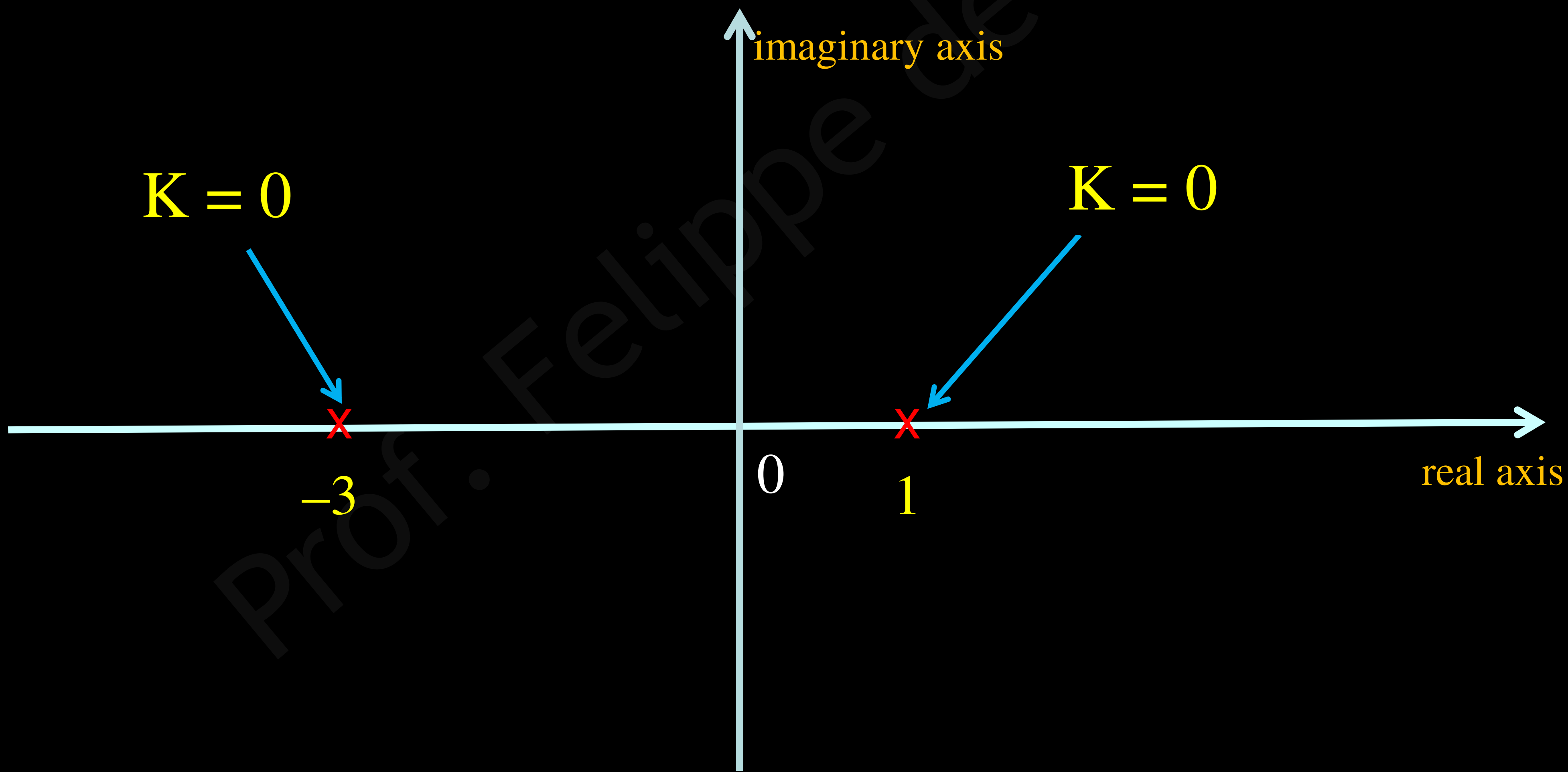
$$\boxed{s = -1 \pm \sqrt{4-K}}$$

in the *complex plane*, when K varies, and for $K > 0$.

Example 3 (*continued*)

Note that:

if $K = 0$ ⇒ $s = -3$ e $s = 1$

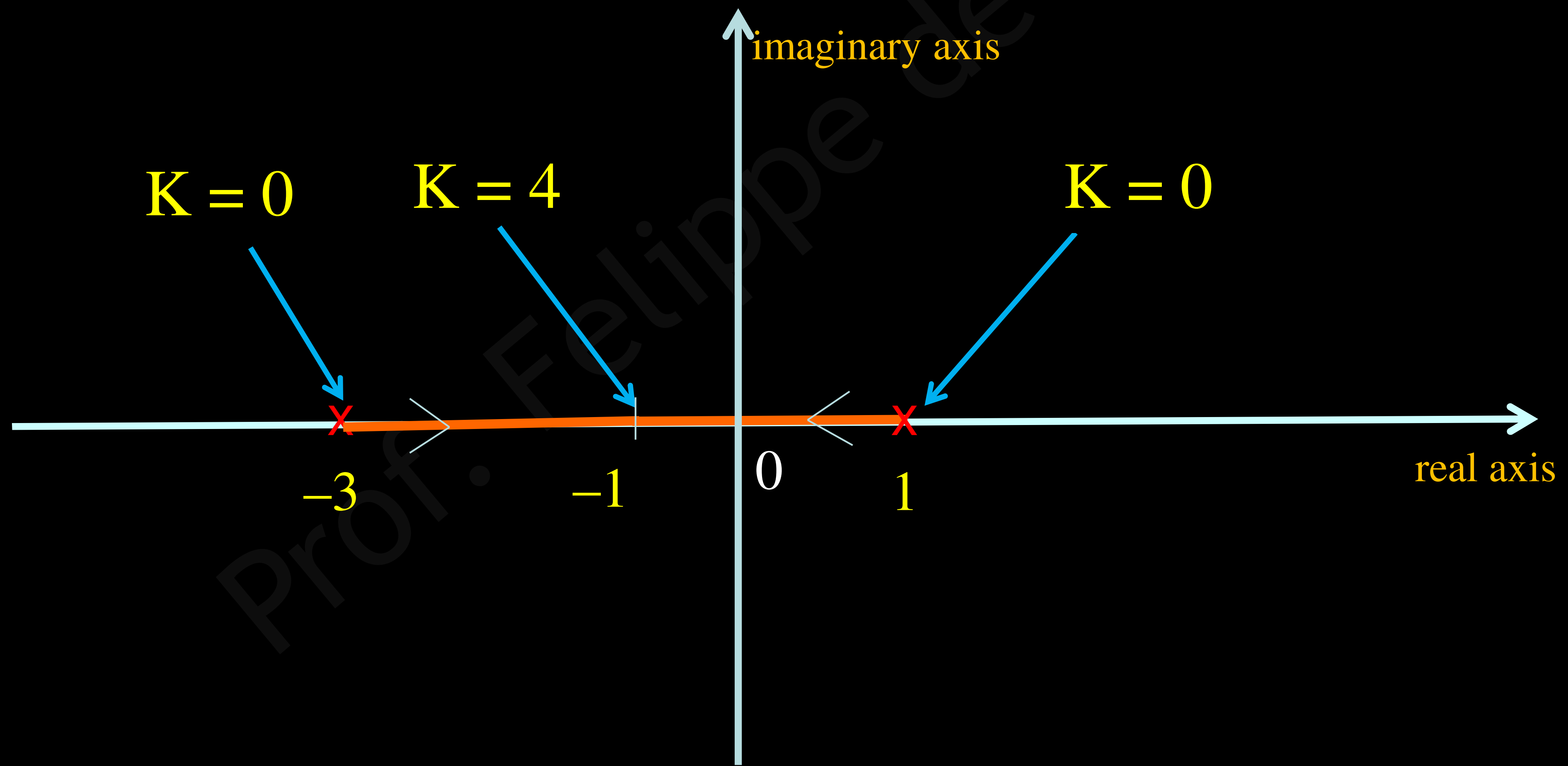


Example 3 (*continued*)

moreover:

if $K < 4$ $\Rightarrow$ $\boxed{s = -1 \pm \sqrt{4 - K}}$ real poles

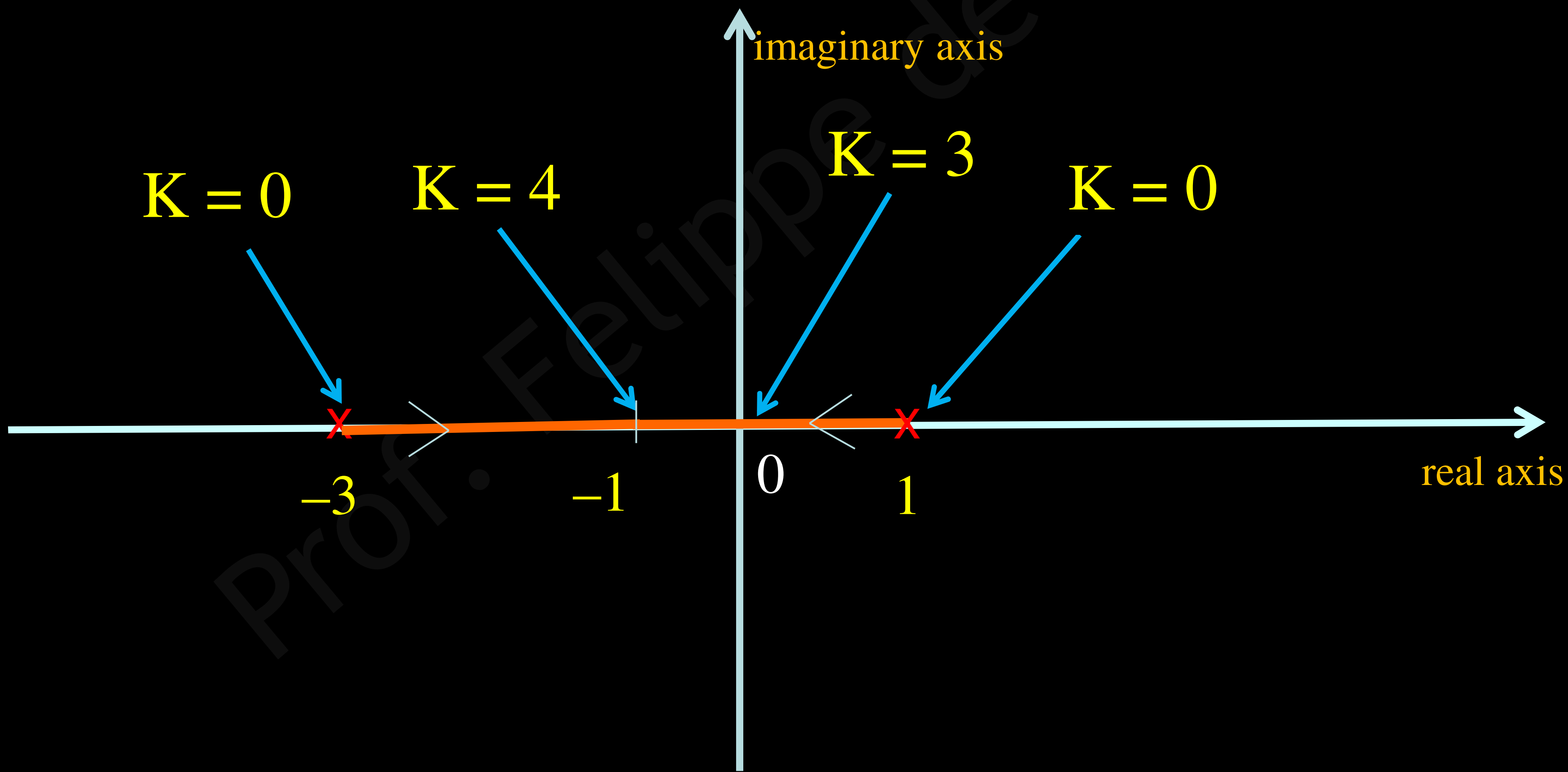
if $K = 4$ $\Rightarrow$ $\boxed{s = -1}$ double real poles



Example 3 (*continued*)

and also:

if $K = 3$ ⟹ $\boxed{s = 0}$ the Root Locus crosses the origin

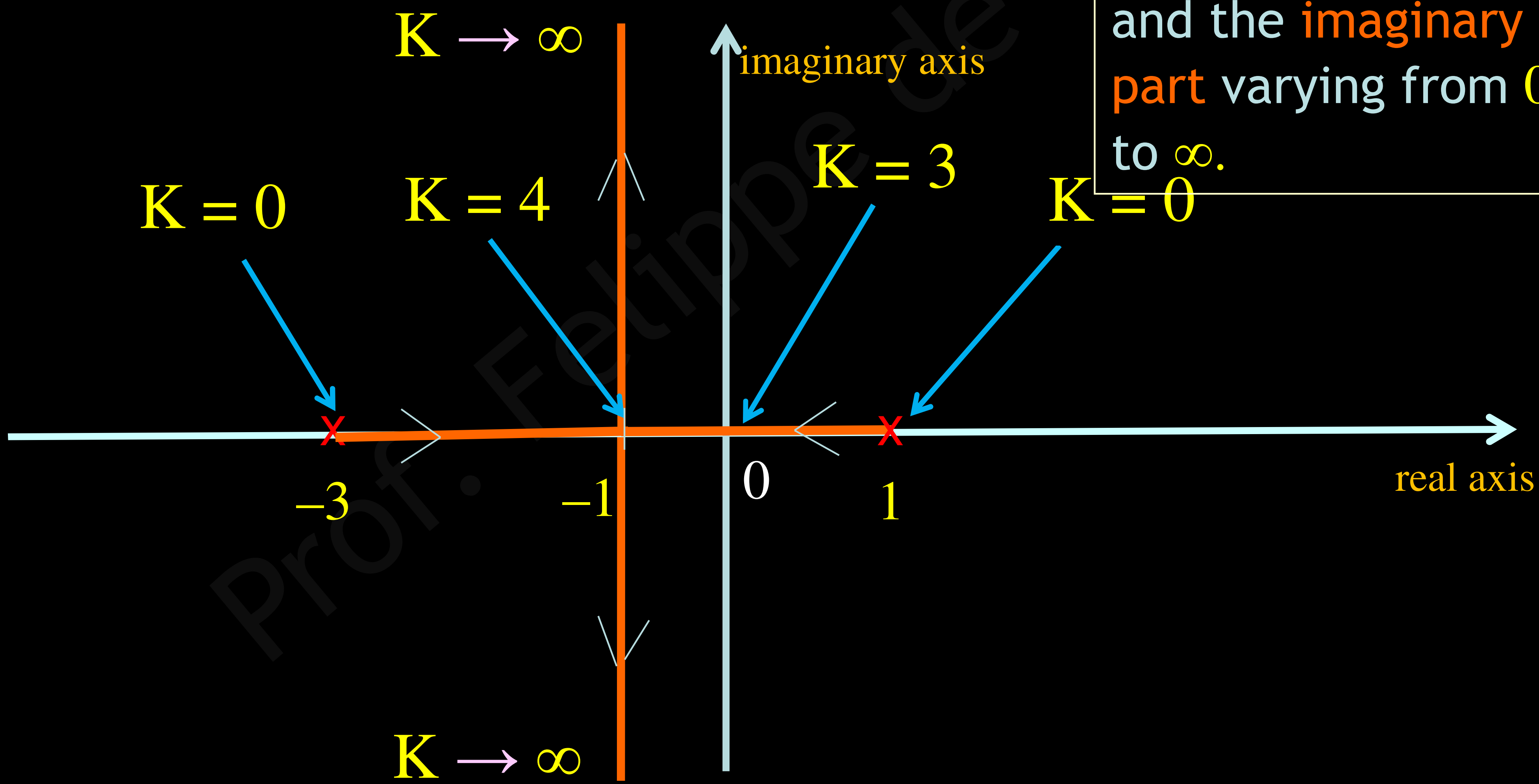


Example 3 (*continued*)

But this Root Locus is not restrict to the real axis, since

if $K > 4$ ⟹ $s = -1 \pm j\sqrt{K-4}$ complex conjugates

with real part = –1 and the imaginary part varying from 0 to ∞.

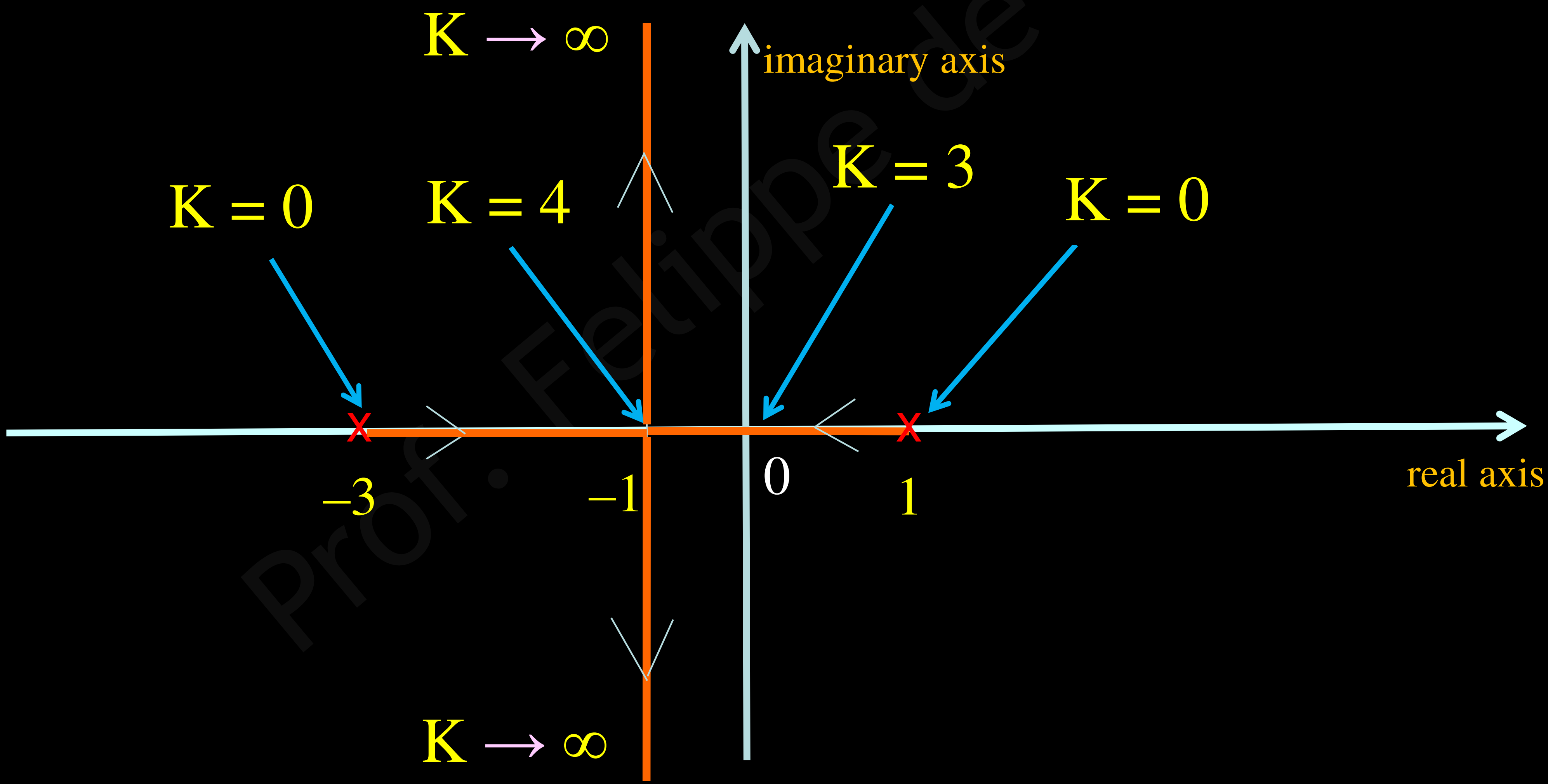


Example 3 (*continued*)

Summarizing, …

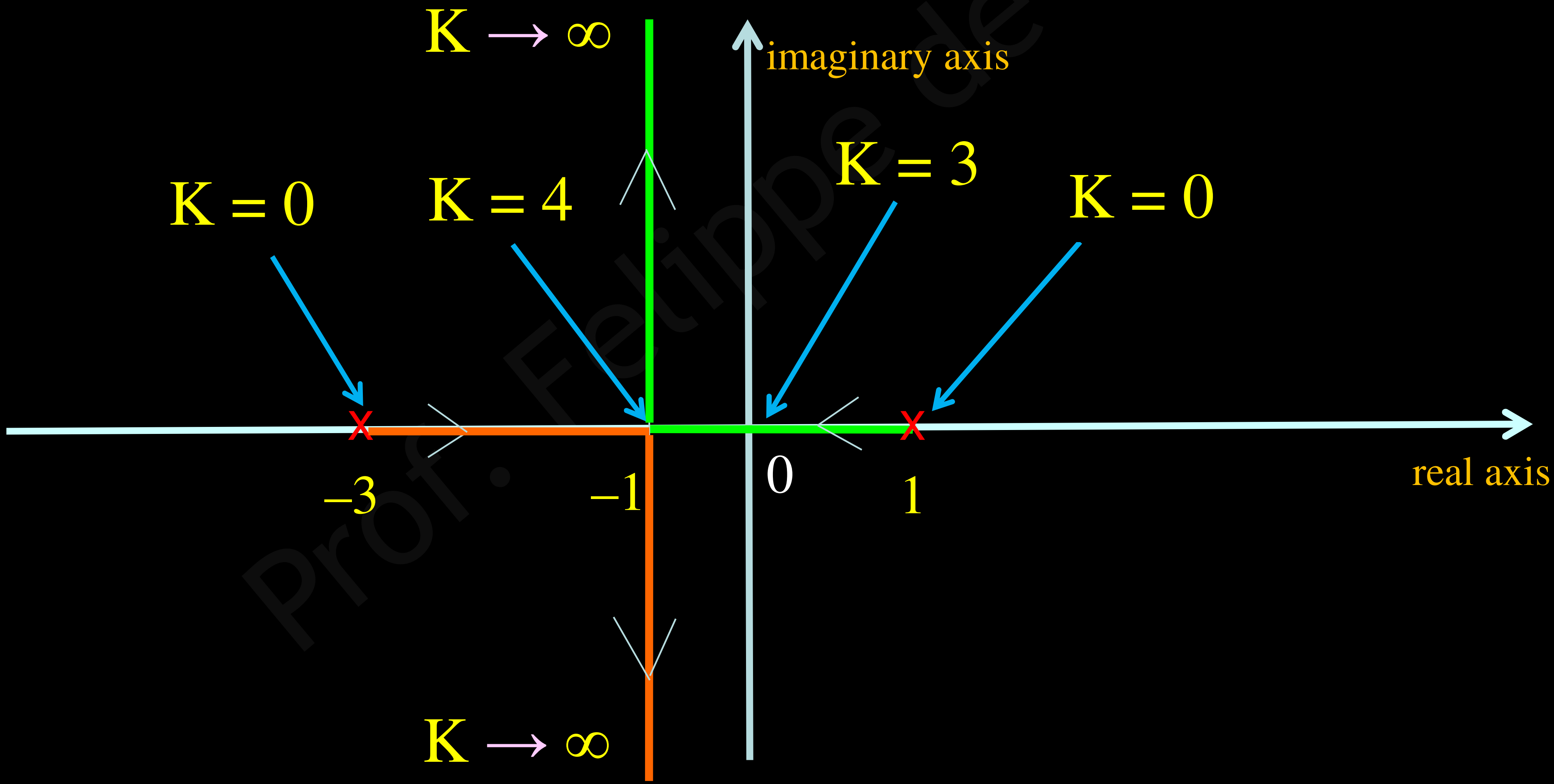
This Root Locus has 2 branches

and now let us present the 2 branches of this Root Locus with *different colours*, as the Matlab does…



Example 3 (*continued*)

and now let us present the 2 branches of this Root Locus with *different colours*, as the Matlab does...



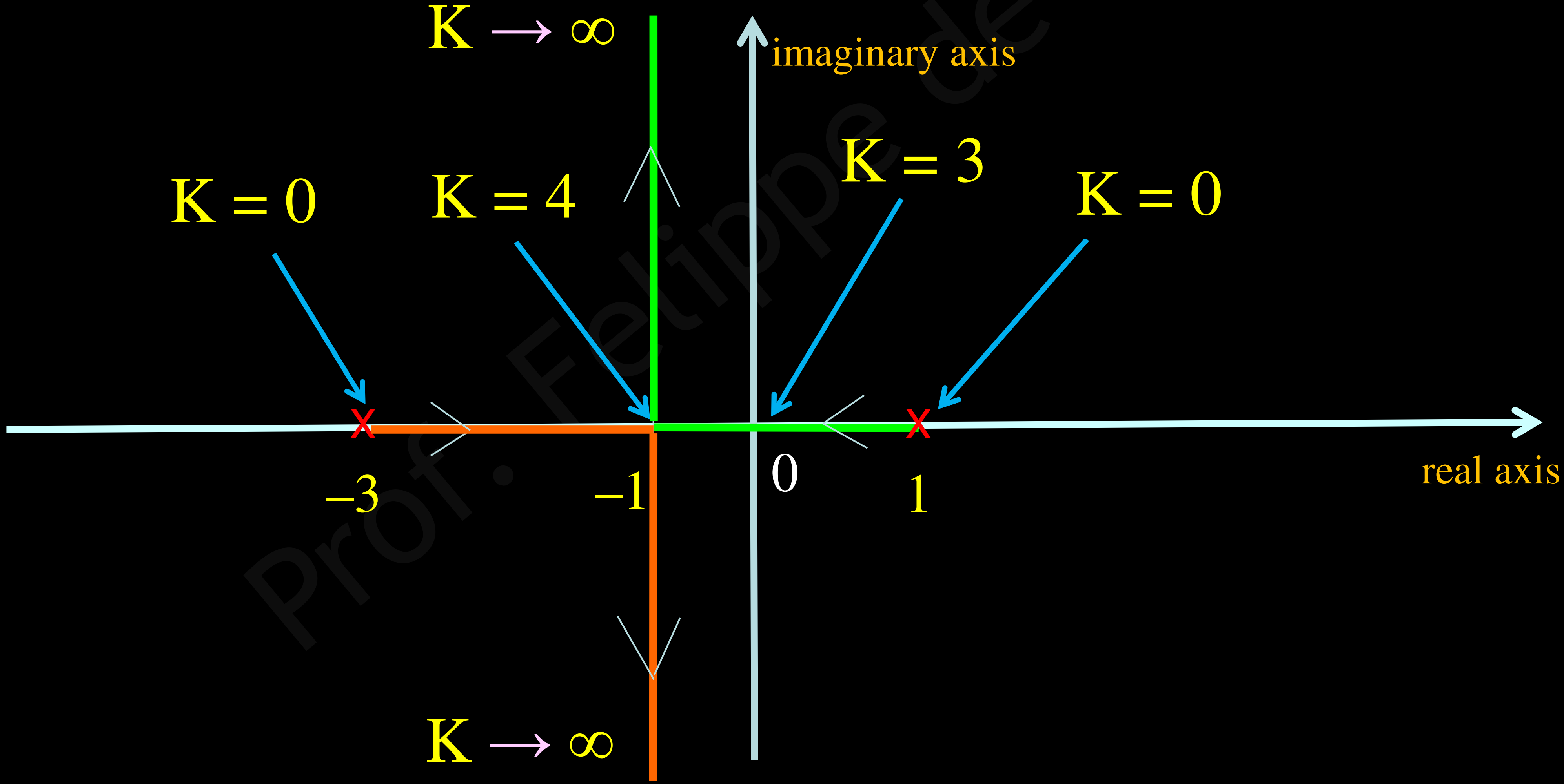
Example 3 (*continued*)

The Root Locus allows us to see that $K \leq 3 \Rightarrow$

$\Rightarrow$ the *closed loop* system is not stable,

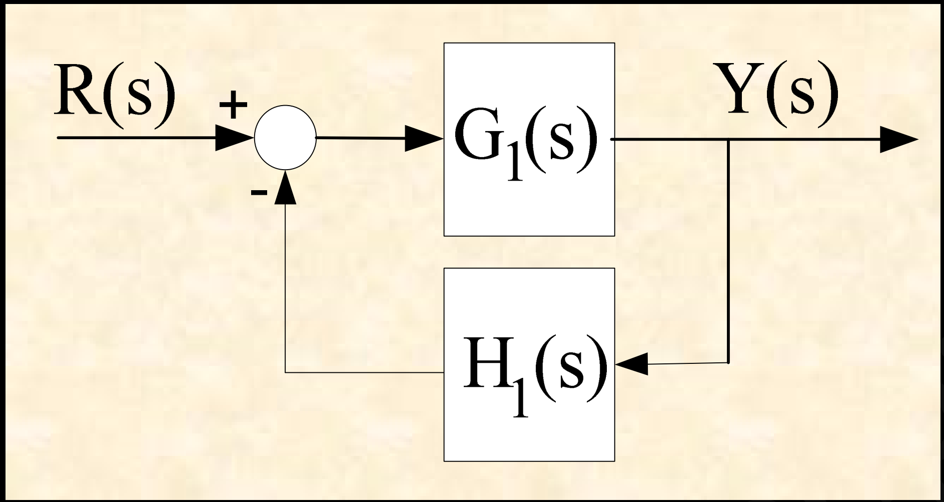
Whereas for $K > 3 \Rightarrow$ the *closed loop* system is stable,

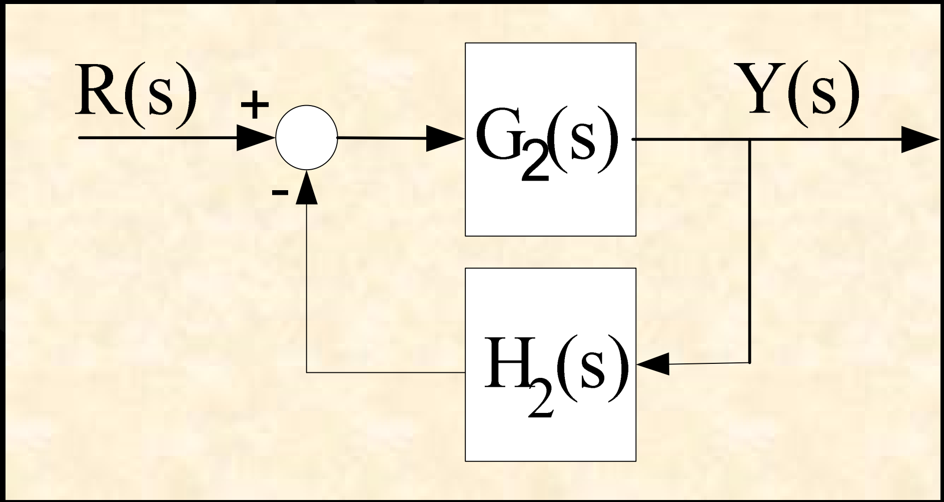
since in that case the 2 *closed loop* poles will lie in the LHP



Note that the "*Root Locus*" depends only on the product $G(s) \cdot H(s)$ and not on $G(s)$ or on $H(s)$ separately

Hence, if the 2 *closed loop systems* below





satisfy

$$G_1(s) \cdot H_1(s) = G_2(s) \cdot H_2(s)$$

Then the "*Root Locus*" of these 2 *systems* are the same

The expression

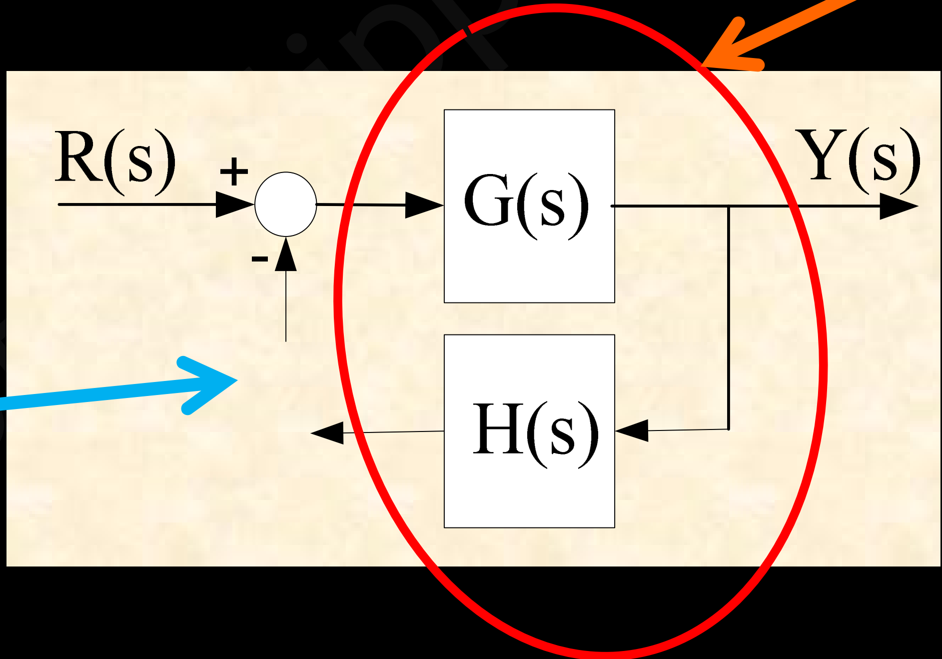
$$G(s)\cdot H(s)$$

Is called the <u>*transfer function*</u> of the *system* in *open loop* (OLTF).

$G(s)\cdot H(s)$

(FTMA)

It is as if the loop had been broken here, becoming open



Therefore, the *poles* and *zeroes* of

$$G(s)\cdot H(s)$$

are called the *open loop* *poles* and *zeroes*

Let us call by n and m the number of *poles* and *zeroes* of $G(s)\cdot H(s)$, respectively

that is:

n = the number of *open loop* *poles*

m = the number of *open loop* *zeroes*

Rules for the construction of the "*Root Locus*"

We shall present here 8 rules that will be very helpful to draw the "*Root Locus*" for a *close loop system* with OLTF given by $G(s) \cdot H(s)$.

Rule #1

Number of branches

Rule #1 – Number of branches

The number of *branches* n of a "*Root Locus*" is the number of open loop *poles*, that is, the number of *poles* of $G(s)\cdot H(s)$.

$$\boxed{n \;=\; \text{n° } \textit{branches} \;=\; \text{n° } \textit{poles} \text{ of } G(s)\cdot H(s)}$$

Rule #2

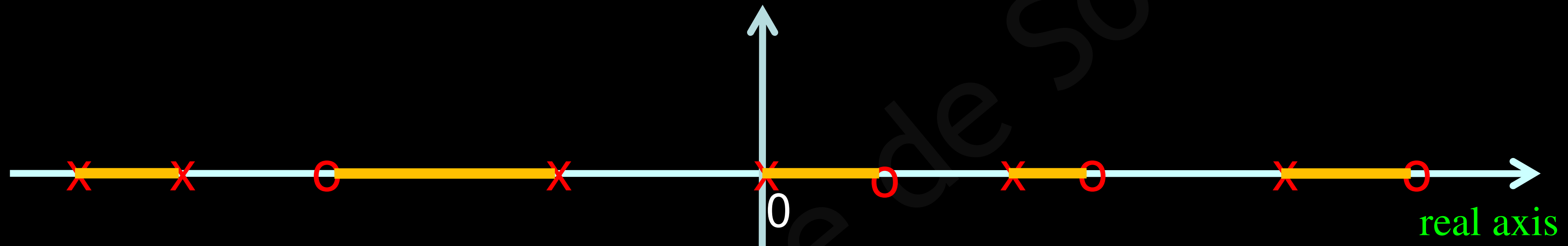
Intervals with and without "*Root Locus*" in the *real axis*

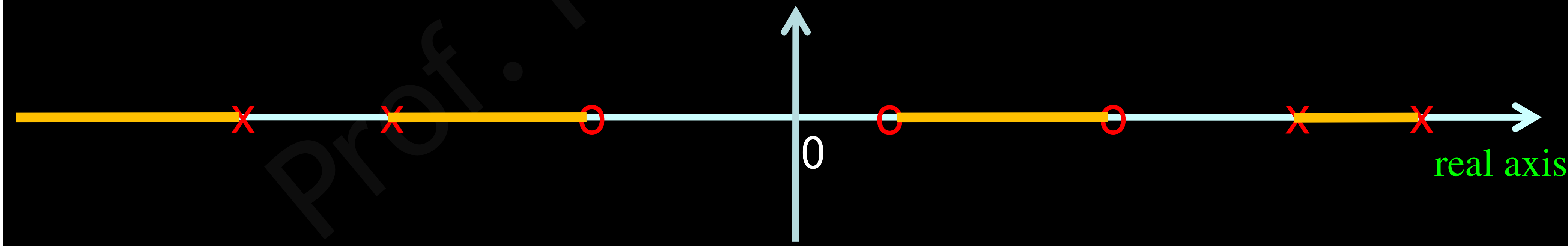
Rule #2 – Intervals with and without "*Root Locus*" in the *real axis*

A point s in the *real axis* belongs to the "*Root Locus*" if there is an odd number of *open loop poles* and *zeroes* to the right of s

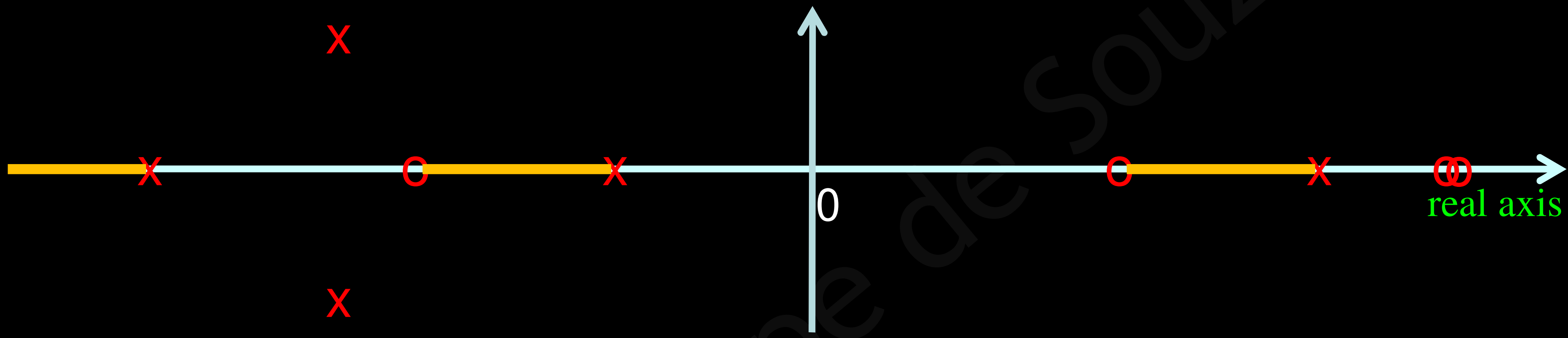
that is, if there is an odd number of *poles* and *zeroes* of $G(s) \cdot H(s)$ to the right of s.

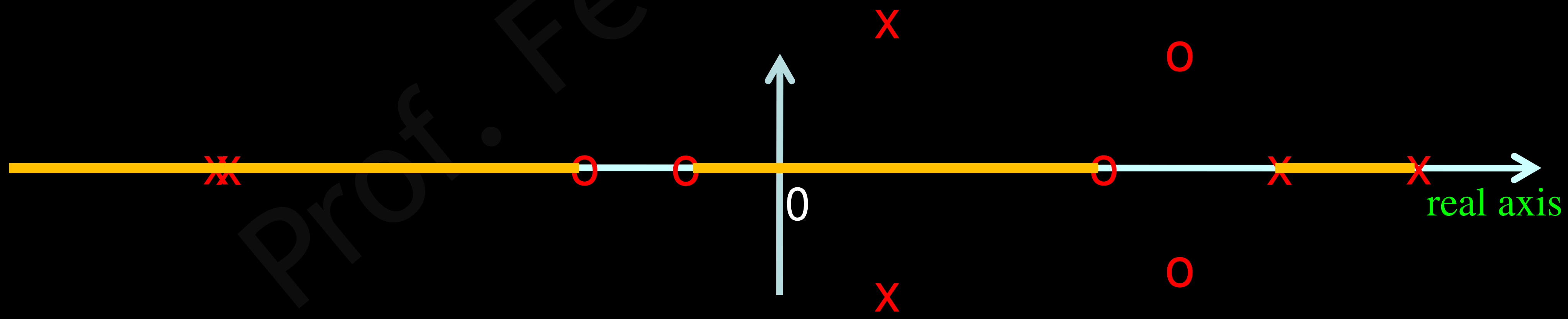
Example 4: Application of Rule #2 – Intervals with and without "*Root Locus*" in the *real axis*





Example 4 (*continued*)   Application of Rule #2

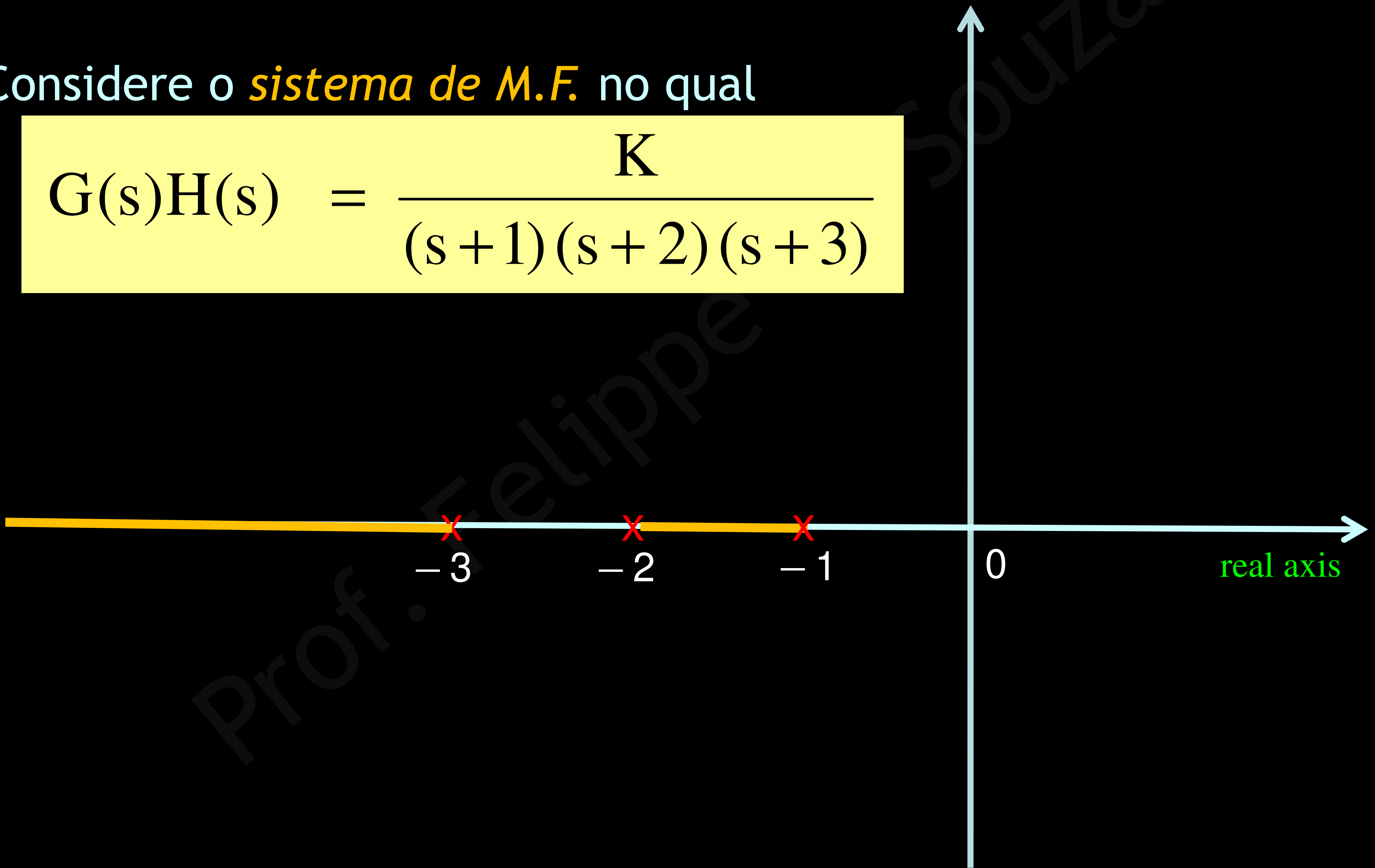




Example 5: Application of Rule #2 – Intervals with and without "*Root Locus*" in the *real axis*

Considere o *sistema de M.F.* no qual

$$G(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)}$$



Rule #3

Beginning and ending points of the branches of the "*Root Locus*"

Rule #3 - Beginning and ending points of the branches of the "*Root Locus*"

The n branches of the "*Root Locus*" start in the
n open loop *poles*
that is, they start in the n *poles* of $G(s) \cdot H(s)$

m of the n branches of the "*Root Locus*" end in the m open loop *zeroes*
that is, they end in the m *zeroes* of $G(s) \cdot H(s)$

and the remainders:
$(n - m)$ branches of the "*Root Locus*" end in *infinite* (∞)

Rule #3 - Beginning and ending points of the branches of the "*Root Locus*"

Summarizing:

$$\begin{cases} n \text{ branches} \rightarrow \text{start at the } n \text{ open loop } \textit{poles} \\ m \text{ branches} \rightarrow \text{end at the } m \text{ open loop } \textit{zeros} \\ (n - m) \text{ branches} \rightarrow \text{end at } \textit{infinity } (\infty) \end{cases}$$

Rule #3 - Beginning and ending points of the branches of the "*Root Locus*" (*continued*)

Note that $(n - m)$ is the difference between the *number of open loop poles* n and the *number of open loop zeroes* m i.e., the difference between the *number of poles* and *zeroes* of $G(s) \cdot H(s)$

If $n = m$ then $(n - m) = 0$, and therefore no *branch* ends in *infinite* (∞)

Hence, if the *number of open loop poles* is equal to the *number of open loop zeroes*, then no *branch* ends in *infinite* (∞)

Rule #4

Asymptotes in the infinite

Rule #4 – Asymptotes in the infinite

For the $(n - m)$ branches of the "*Root Locus*" that do not end in the m *open loop zeroes*, that is, the m *finite zeroes* of $G(s) \cdot H(s)$, we can determine the direction which they are going to *infinite* in the *complex plane*

γ = angle of the asymptote with the *real axis*

$$\gamma = \frac{180^o \cdot (2i + 1)}{(n - m)} \qquad i = 0, 1, 2, \cdots$$

Rule #4 – Asymptotes in the infinite (*continued*)

Applying the formula we obtain the table below:

$n - m$	γ = angle of the asymptote with the *real axis*
1	180º
2	90º and –90º
3	60º, –60º and 180º
4	45º, –45º, 135º and –135º
5	36º, –36º, 108º, –108º and 180º
6	30º, –30º, 90º, –90º, 150º and –150º
⋮	⋮ ⋮

Rule #5

Interception points of the asymptotes with the *real axis*

Rule #5 - Interception points of the asymptotes with the *real axis*

The $(n - m)$ asymptotes in the infinite are well determined by its directions (*angles* γ) and by the *point* where they meet in the *real axis*, σ_o given by the expression:

$$\sigma_o = \frac{\left(\sum_{i=1}^{n} \mathrm{Re}(p_i) \; - \; \sum_{j=1}^{m} \mathrm{Re}(z_j) \right)}{(n - m)}$$

Rule #6

Points in the *real axis* where there are *branch encounters*

Rule #6 – Points in the *real axis* where there are *branch encounters*

Firstly we write the equation

$$1 + G(s)\cdot H(s) = 0,$$

and then we obtain an expression for K as a function of s:

$$K(s)$$

thus, we calculate the derivative of K with respect to s, dK/ds

Now, using the equation below which depends on s

$$\boxed{\frac{dK}{ds} = 0}$$

We obtain the points s in the <u>*real axis*</u> where there are *meetings* of *branches*

Rule #6 – Points in the *real axis* where there are *branch encounters* (*continued*)

This equation in s

$$\boxed{\frac{dK}{ds} = 0}$$

May have a number of solutions

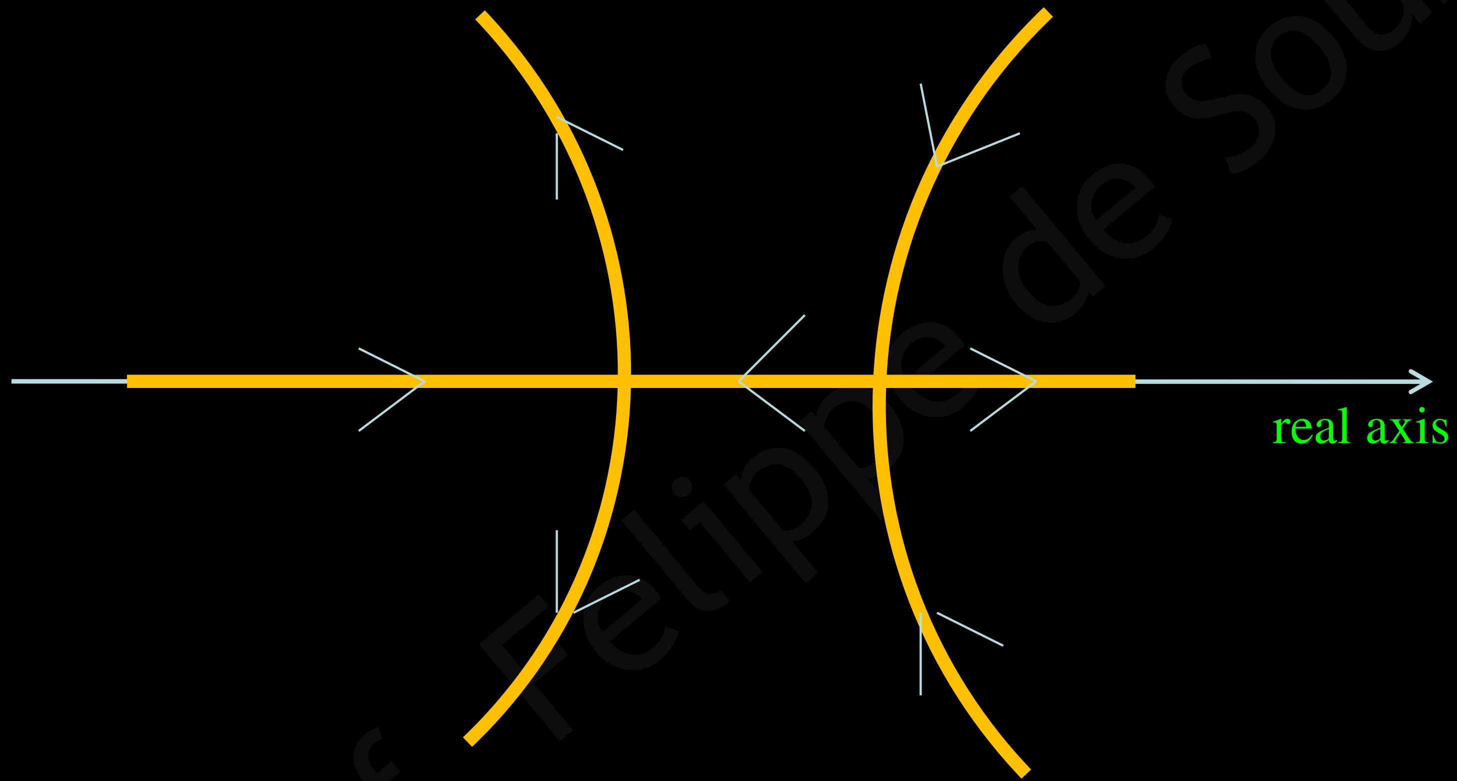
$s = s_1 \qquad s = s_2 \qquad s = s_3 \qquad s = s_4 \qquad s = s_5 \quad \ldots \ldots \ldots$

which are greater than the number of *points where branches meet* in the *real axis*

It will be necessary to cancel solutions that are not points belonging to the "*Root Locus*"

$s = s_1 \qquad \cancel{s = s_2} \qquad s = s_3 \qquad \cancel{s = s_4} \qquad s = s_5 \quad \ldots \ldots \ldots$

Rule #6 – Points in the *real axis* where there are *branch encounters* (*continued*)



When there are *branches encounter* in the <u>*real axis*</u>, it may be *branches* that *meet* and <u>ENTER</u> the <u>*real axis*</u> or *branches* that *meet* and <u>LEAVE</u> the <u>*real axis*</u>

Rule #6 – Points in the *real axis* where there are *branch encounters* (*continued*)

For a point

$$s = s'$$

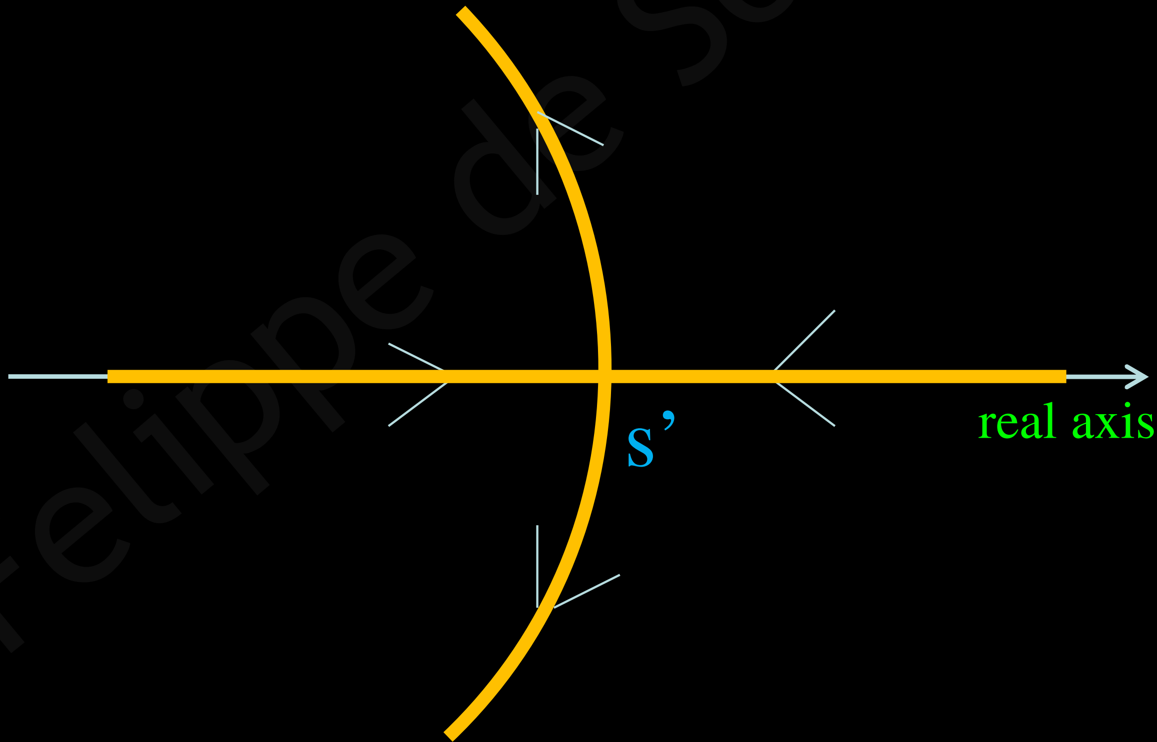
where there are *branch* *encounters* in the *real axis*,

we calculate the *second derivative* of $K(s)$ at $s = s'$

$$\left.\frac{d^2K}{ds^2}\right|_{s=s'}$$

Rule #6 – Points in the *real axis* where there are *branch encounters* (*continued*)

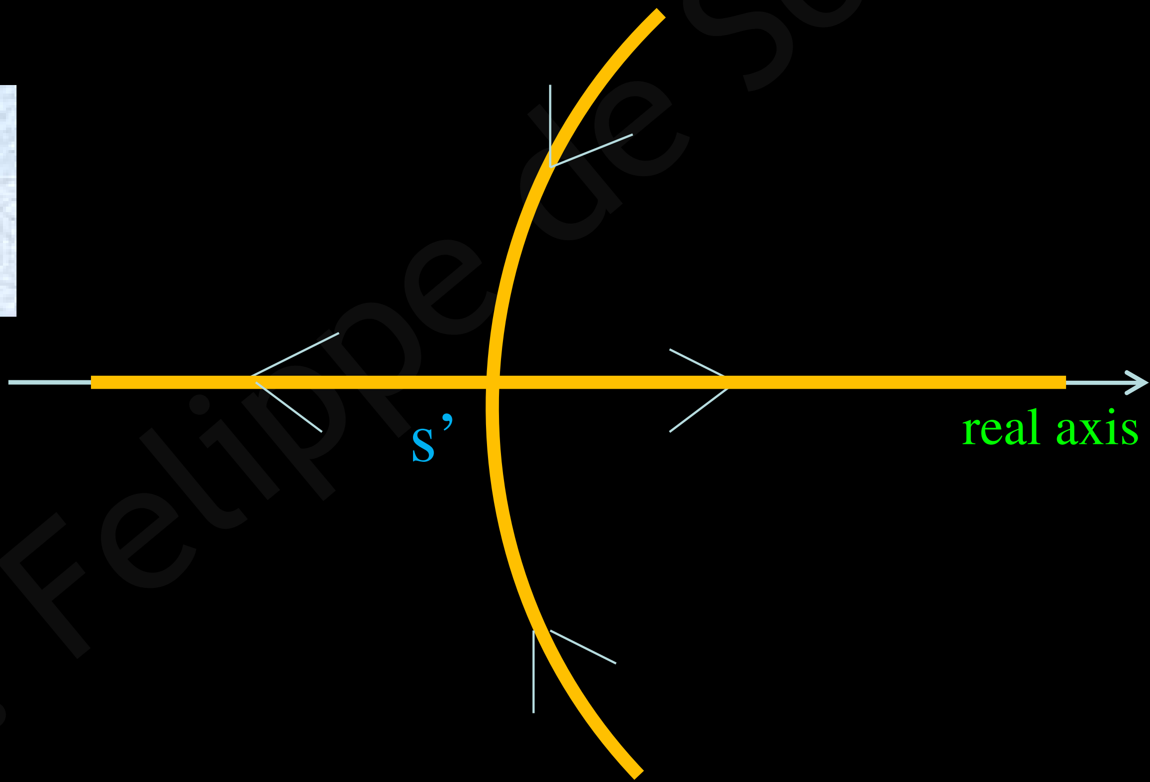
If $\left.\dfrac{d^2K}{ds^2}\right|_{s=s'} < 0$



$\Rightarrow$ 2 *branches* that *meet* and LEAVE the *real axis*

Rule #6 – Points in the *real axis* where there are *branch encounters* (*continued*)

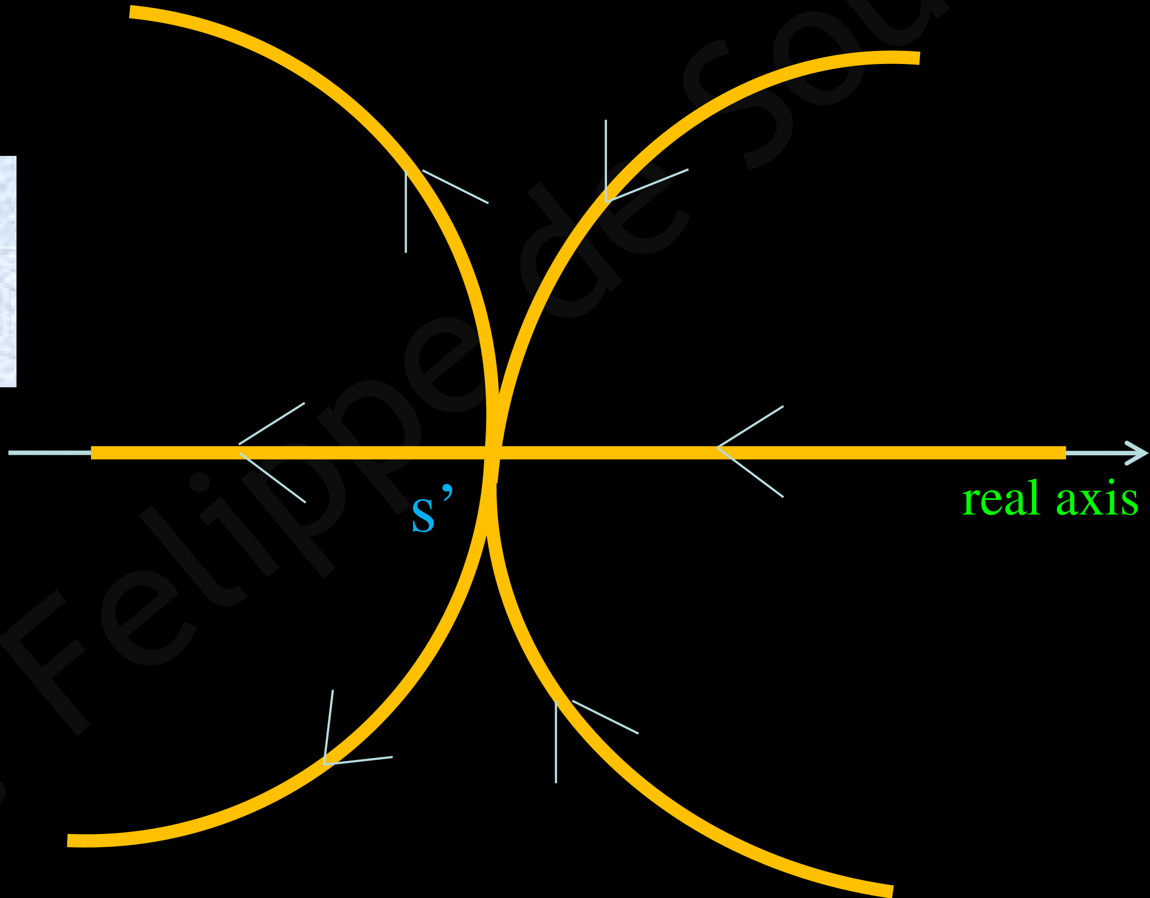
If $\left.\dfrac{d^2K}{ds^2}\right|_{s=s'} > 0$



$\Rightarrow$ 2 *branches* that *meet* and ENTER the *real axis*

Rule #6 – Points in the *real axis* where there are *branch encounters* (*continued*)

If $$\left.\frac{d^2K}{ds^2}\right|_{s=s'} = 0$$



⇒ more than 2 *branches* that *meet* in this point

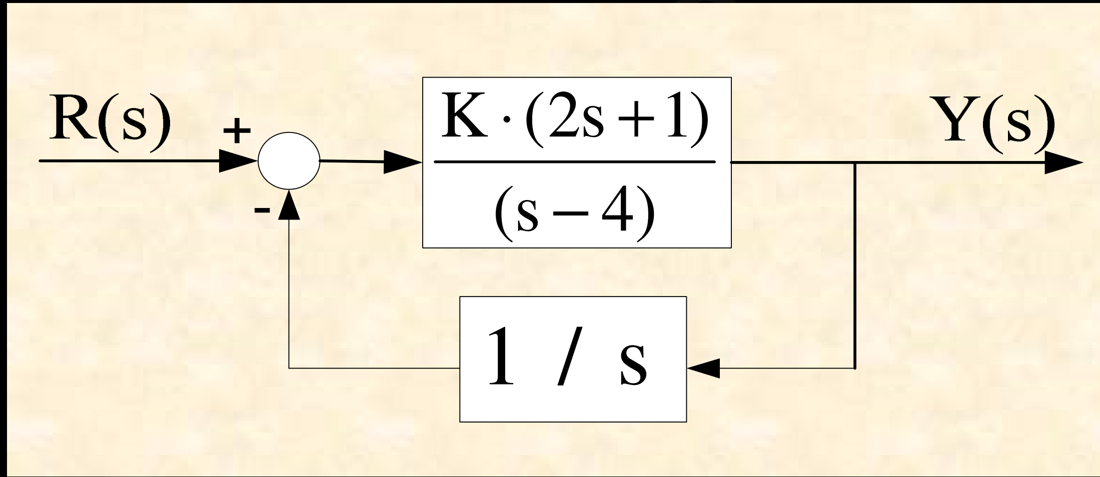
⇒ we follow to Rule #7 – Encounter of more than two branches

Example 6: Application of Rule #6 – Points in the *real axis* where there are *branch encounters*

Returning to Example 1, let

$$1 + G(s)H(s) = $$

$$= 1 + \frac{K \cdot (2s+1)}{(s-4)\,s} = 0$$



then we have:

$$K = -\frac{(s-4)\,s}{(2s+1)}$$

and therefore:

$$\frac{dK}{ds} = \frac{2\,(s^2+s-2)}{(2s+1)^2} = 0 \quad \Rightarrow \quad \begin{cases} s = 1 \\ s = -2 \end{cases}$$

thus, $\underline{s = 1}$ and $\underline{s = -2}$ are the *points* where there are *branches meeting*

Example 6 (*continued*) Application of Rule #6

In order to know the value of K in each of these *points*
It is necessary to substitute ($s = 1$ e $s = -2$) in the *expression* of K

$$K = -\left.\frac{(s-4)s}{(2s+1)}\right|_{s=1} = 1$$

$$K = -\left.\frac{(s-4)s}{(2s+1)}\right|_{s=-2} = 4$$

Thus, the *points* where there are *encounter* of *branches* are

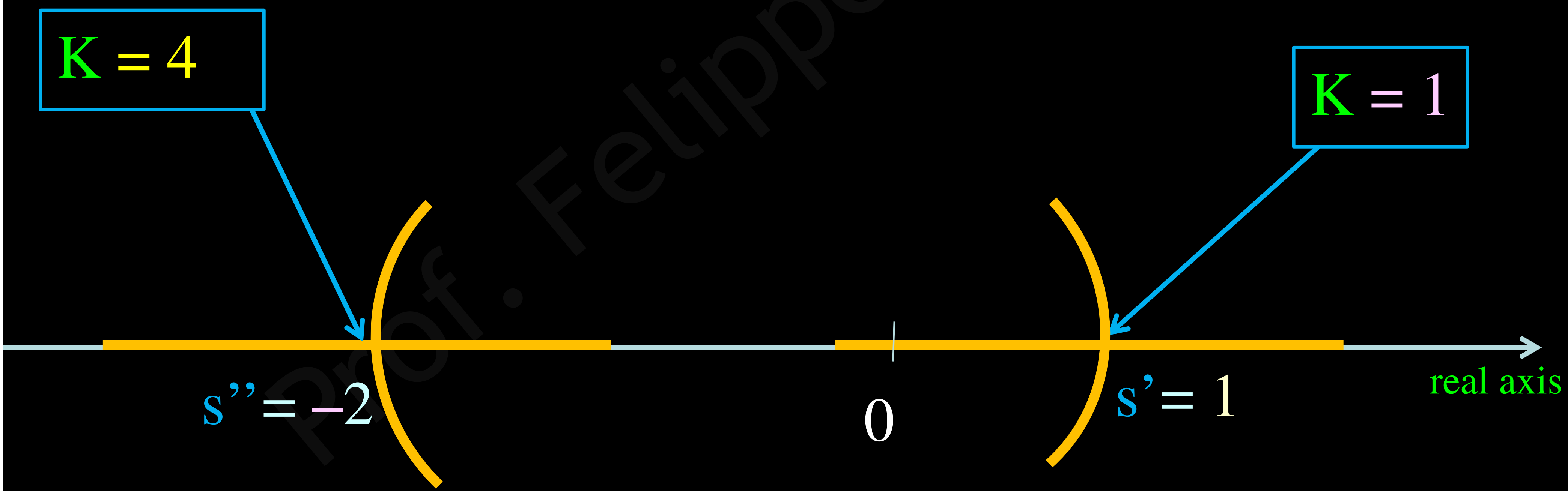
$$\Rightarrow \begin{cases} s = 1 \;\; (K = 1) & \text{and} \\ s = -2 \;\; (K = 4) & \end{cases}$$

Example 6 (*continued*)   Application of Rule #6

the *points* where there are *meting* of *branches* are

$s = 1$ $(K = 1)$ and

$s = -2$ $(K = 4)$



Example 6 (*continued*) Application of Rule #6

Now, in order to find if each of these *branch encounters* are ENTERING *branches* or LEAVING *branches*, it is necessary to calculate the second derivative

$$\frac{d^2K}{ds^2} = -\frac{18}{(2s+1)^3}$$

substituting by the *points* where *branches meet*:

$s = 1$ and $s = -2$

$$\left.\frac{d^2K}{ds^2}\right|_{s=1} = \left.\frac{-18}{(2s+1)^3}\right|_{s=1} = -\frac{2}{3} < 0$$

→ *branches* that LEAVE the *real axis*

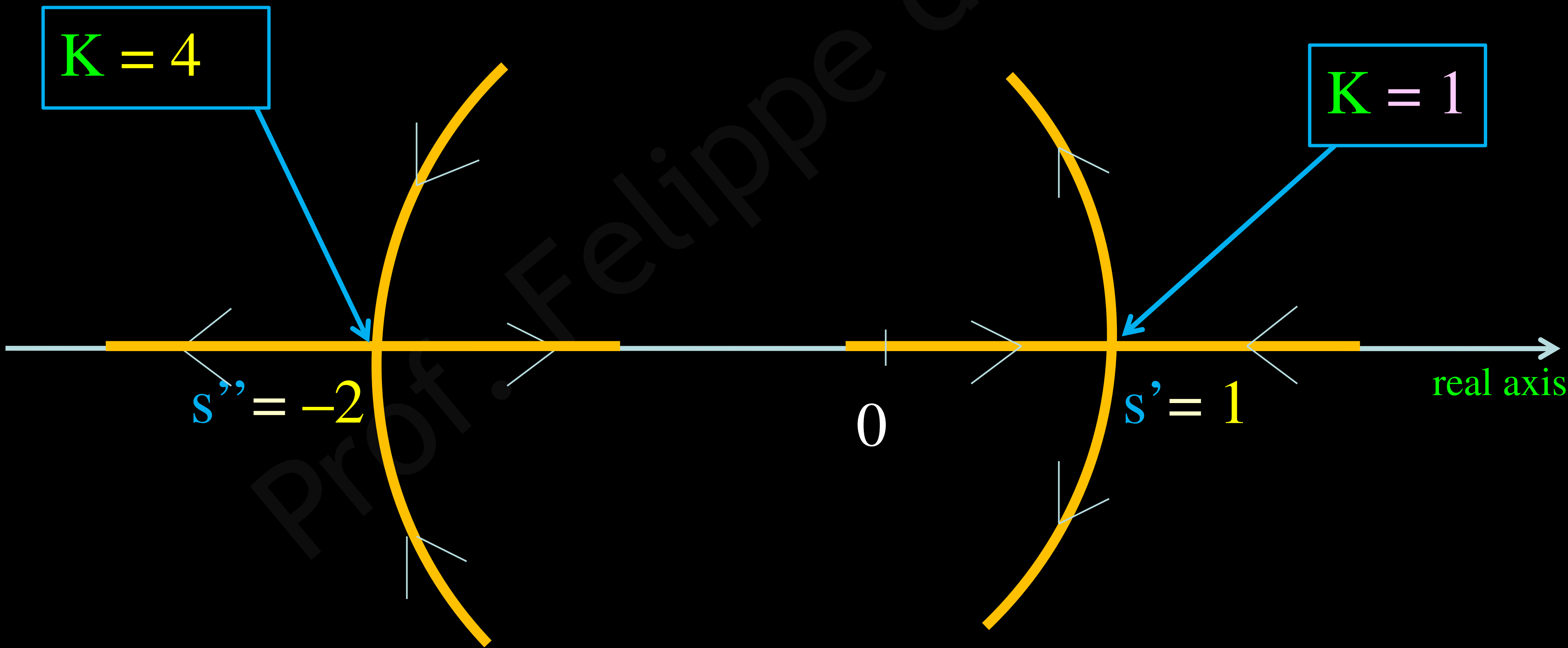
$$\left.\frac{d^2K}{ds^2}\right|_{s=-2} = \left.\frac{-18}{(2s+1)^3}\right|_{s=-2} = +\frac{2}{3} > 0$$

→ *branches* that ENTER the *real axis*

Example 6 (*continued*) Application of Rule #6

Thus, the points where there are *branches meeting* are:

$$\begin{cases} \underline{s = 1} \ \ (\underline{K = 1}) \text{ branches that LEAVE the real axis} \\ \underline{s = -2} \ \ (\underline{K = 4}) \text{ branches that ENTER the real axis} \end{cases}$$



Example 7: Application of Rule #6 – Points in the *real axis* where there are *branches* meeting

Returning to Example 5

$$G(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

hence

$$1+G(s)H(s) = 1+\frac{K}{(s+1)(s+2)(s+3)} = 0$$

$$\begin{aligned} K &= -(s+1)(s+2)(s+3) \\ &= -s^3 - 6s^2 - 11s - 6 \end{aligned}$$

$$\frac{dK}{ds} = -3s^2 - 12s - 11$$

Example 7 (*continued*) Application of Rule #6

thus, setting

$$\boxed{\frac{dK}{ds} = 0} \Rightarrow \boxed{-3s^2 - 12s - 11 = 0}$$

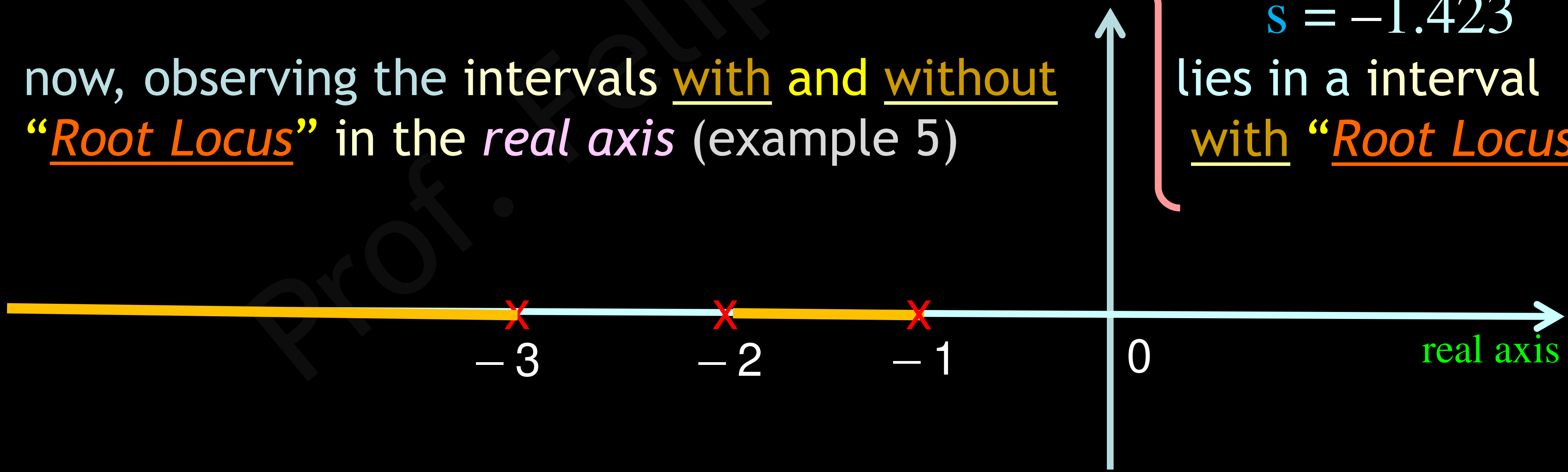
$$\Rightarrow \begin{cases} s = \cancel{-2.58} \\ s = -1.423 \end{cases}$$

We conclude that only one of the solutions,

$$s = -1.423$$

lies in a interval with "*Root Locus*"

now, observing the intervals with and without "*Root Locus*" in the real axis (example 5)



Rule #7

Encounter of more than two branches

Rule #7 – Encounter of more than two branches

When applying the *previous rule*, if

$$\boxed{\left.\frac{d^2K}{ds^2}\right|_{s=s'} = 0}$$

this means that there are encounter of *more than two branches* and we have to keep on the differentiation on $K(s)$, to higher order *derivatives*

$$\boxed{\frac{d^kK}{ds^k}} \qquad \boxed{k = 3, 4, 5, \cdots}$$

until we get

$$\boxed{\left.\frac{d^\eta K}{ds^\eta}\right|_{s=s'} \neq 0}$$ for some η

Rule #7 – Encounter of more than two branches (*continued*)

If $\left.\dfrac{d^{\eta}K}{ds^{\eta}}\right|_{s=s'} \neq 0$ and $\left.\dfrac{d^{k}K}{ds^{k}}\right|_{s=s'} = 0$ for $\forall\ k < \eta$

this means that there is an *encounter* of η *branches* at s'

that is, η branches ENTER and η branches LEAVE at s'

The *meeting* of 3 branches or more it is not much common Certainly occurs with *less frequency* than the *meeting* of 2 branches (Rule #6)

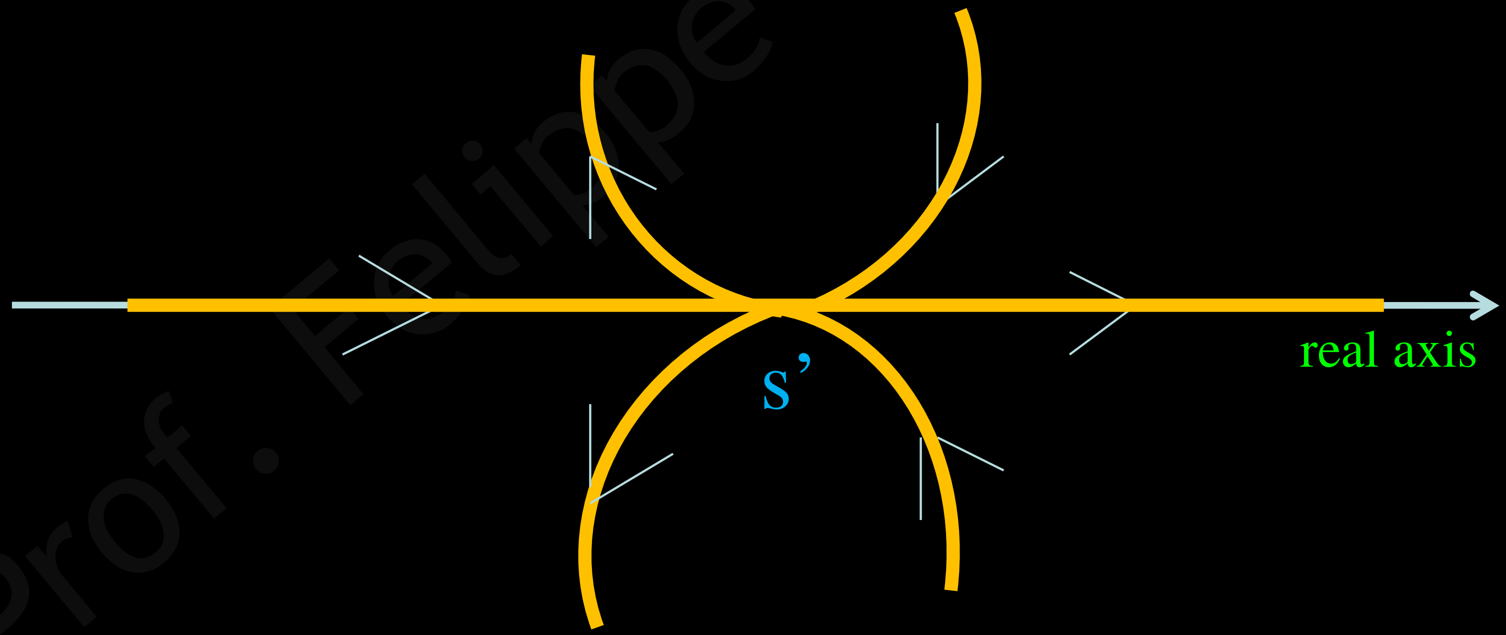
Thus, this Rule #7 is not always used. Only in those cases where, by applying Rule #6, we find

$$\left.\frac{d^{2}K}{ds^{2}}\right|_{s=s'} = 0$$

Rule #7 - Encounter of more than two branches (*continued*)

A *meeting* of 3 *branches* at s' can have the following aspect

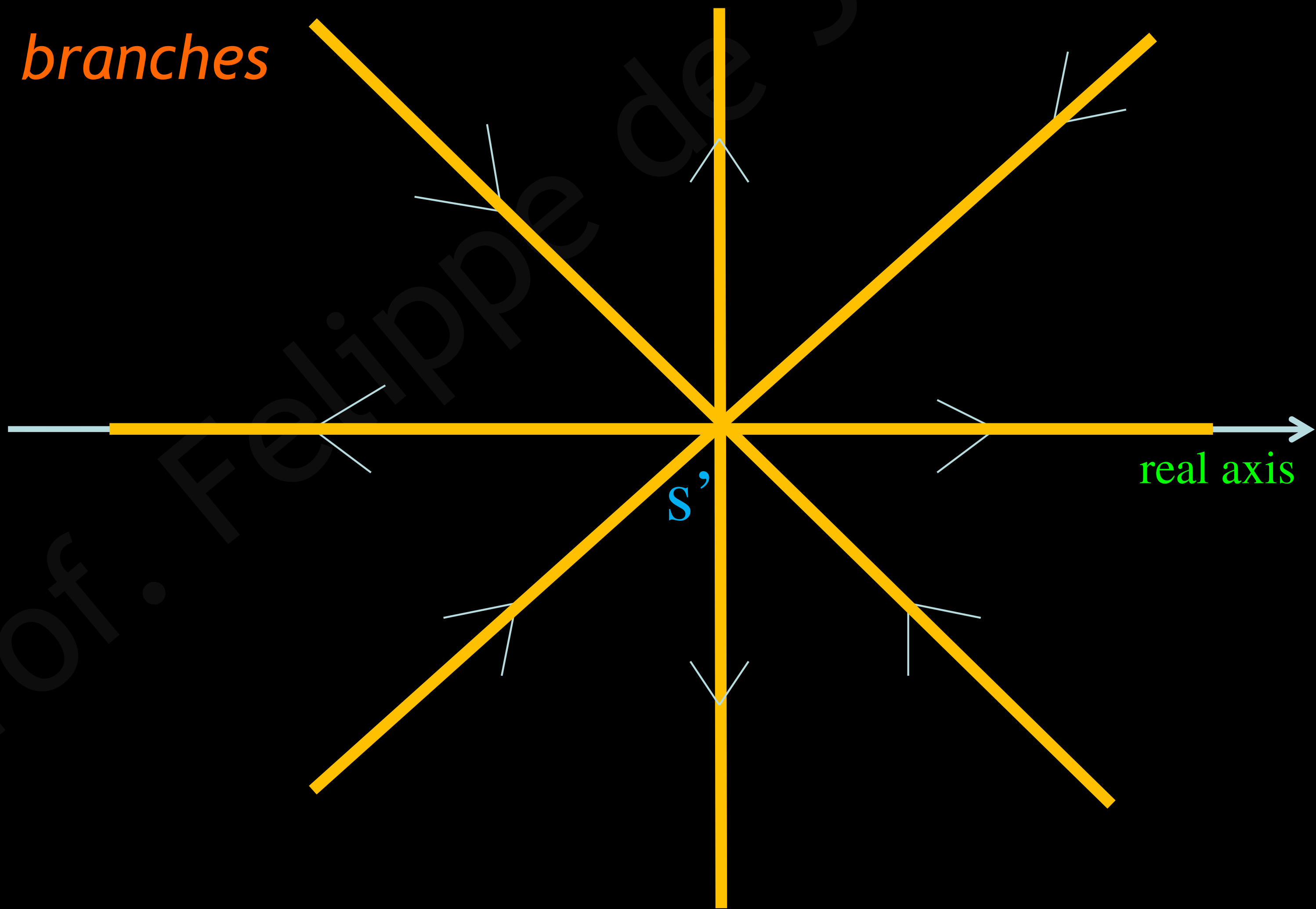
3 ENTERING *branches*
and 3 LEAVING *branches*
at s'



Rule #7 – Encounter of more than two branches (*continued*)

A *meeting* of 4 *branches* at s' can have the following aspect

4 ENTERING *branches*
and 4 LEAVING *branches*
at s'



Example 8: Application of Rule #7 – Encounter of more than two branches

$$G(s)H(s) = \frac{K}{(s^3 - 1)}$$

$$1 + G(s)H(s) = 0 \Rightarrow K = -s^3 + 1$$

$$\Rightarrow \frac{dK}{ds} = -3s^2 = 0 \Rightarrow s' = 0$$

$$\Rightarrow \left.\frac{d^2K}{ds^2}\right|_{s=0} = -6s|_{s=0} = 0 \Rightarrow$$

apply Rule #7

(starting with the 3rd order derivative)

Example 8 (*continued*) Application of Rule #7

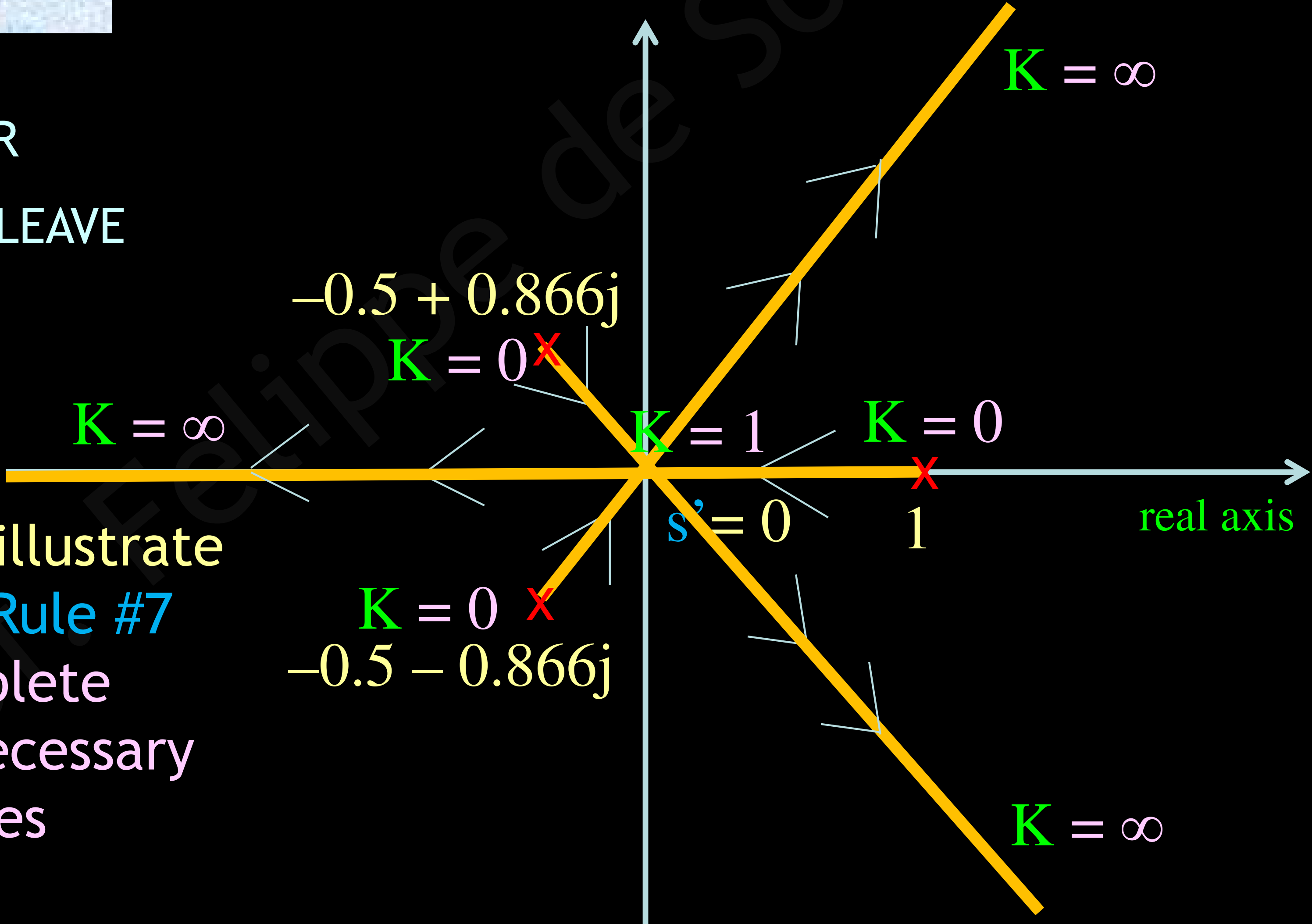
$$\left.\frac{d^3K}{ds^3}\right|_{s=0} = -6 \neq 0$$

meeting of 3 branches at $s' = 0$

3 *branches* ENTER

and 3 *branches* LEAVE

at $s' = 0$



This example only illustrate the application of Rule #7
To sketch the complete "*Root Locus*" it is necessary to apply all the rules

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Obrigado!

Thank you!

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