Control Systems

10

"Stability"

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Stability

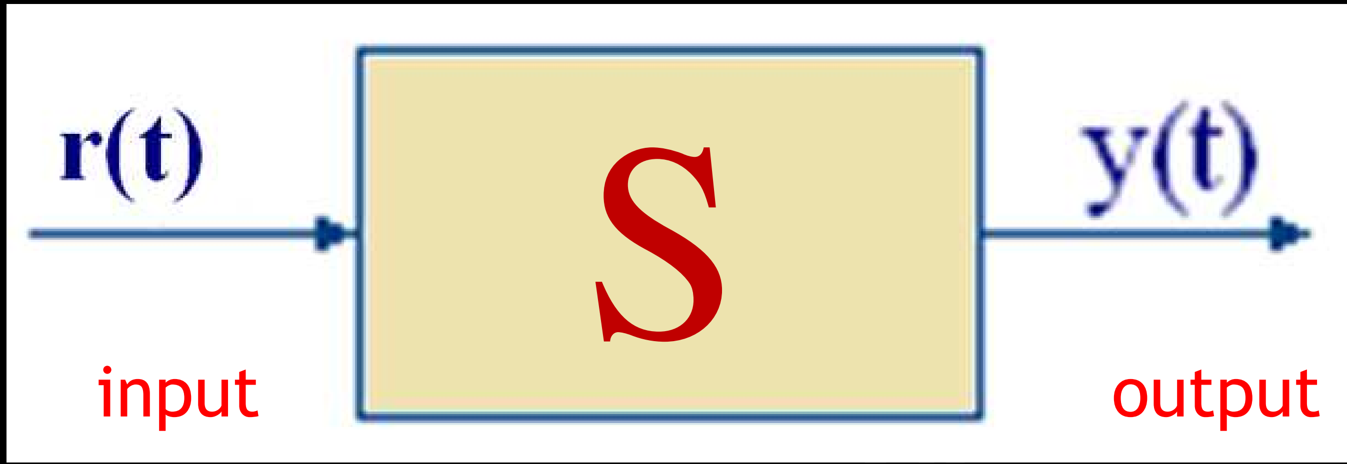
The *system* is stable if the *output* to the impulse *input* $\to 0$ whenever $t \to \infty$

That is, if the *output* of the *system* satisfies

$$\lim y(t) \Big|_{t \to \infty} = 0$$

whenever the input r(t) = impulse

Stability



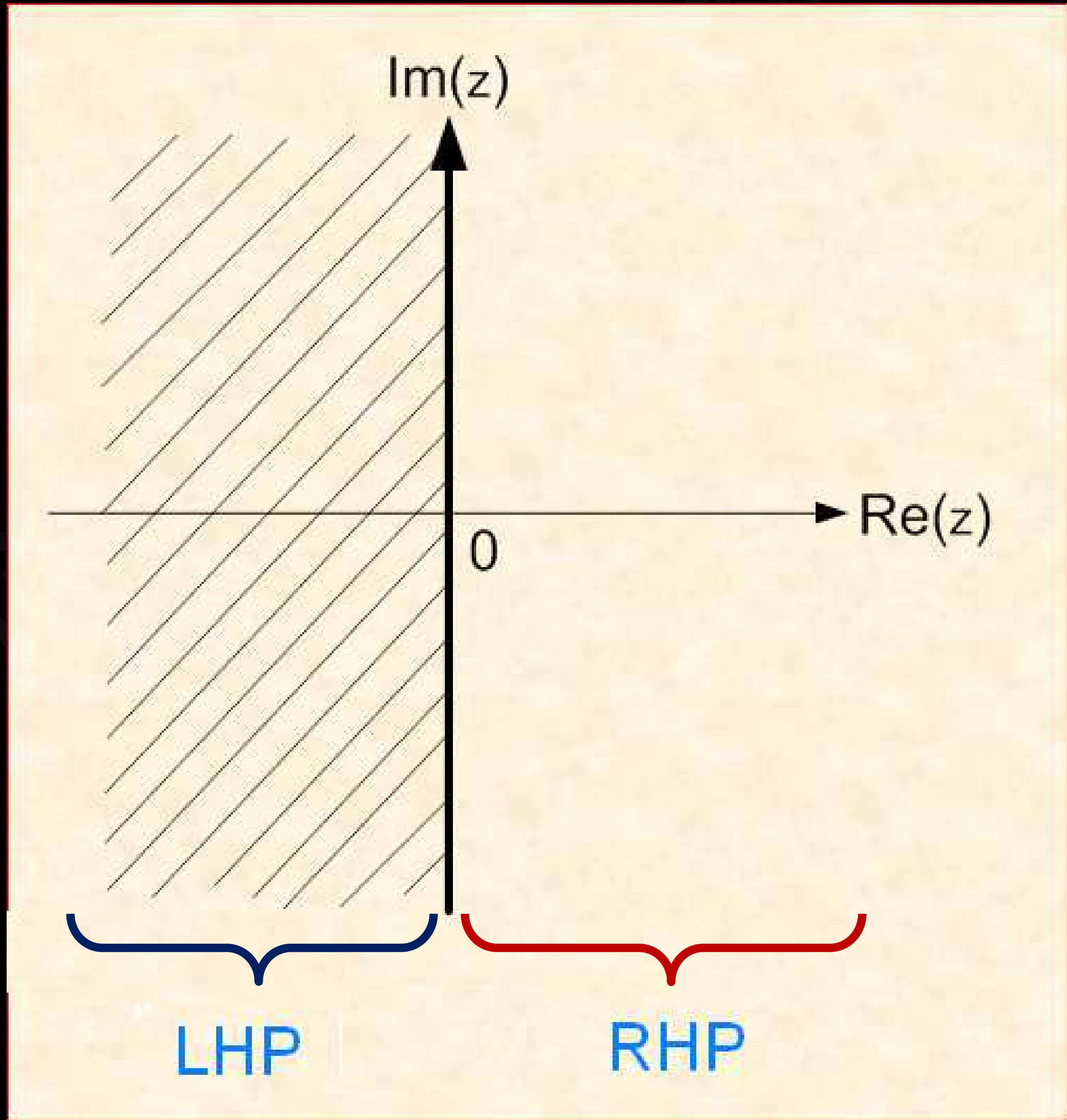
There are other definitions which are equivalent, such as:

The *system* is *stable* if for every limited *input* there is a limited *output*

Due to this definition, *stable systems* are commonly called by BIBO-stable

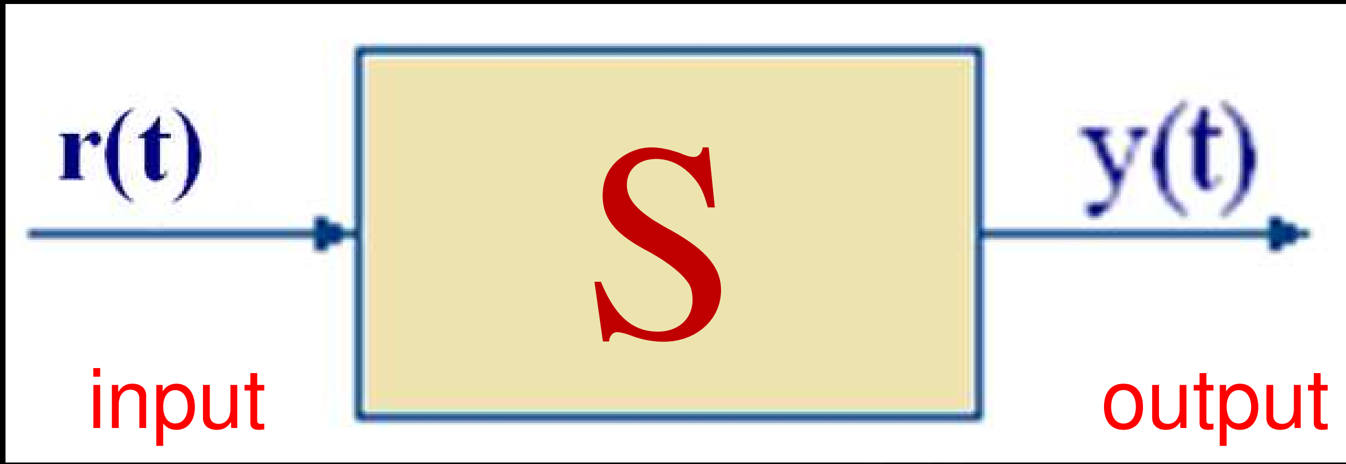
(BIBO = *bounded input-bounded output*)

A *system* is **stable** if, and only if, it has ***all*** its *poles* with *negative real parts*



That is, a *system* is ***stable*** if it has got ***all*** its *poles* located in the *left half plane* (*LHP*)

Stability



So, the *stability* of *systems* can be determined by the *location* of the *poles* of the *system* in the *complex plane*

It is necessary that ALL the *poles* of the *system* lie in the LHP in order to be a *stable system*.

One single *pole* that do not lie in the LHP ruins the *stability* turning it in a *unstable system*.

Example 1:

Consider the de 1st order *system*

$$\begin{cases} x' = a\,x + u \\ y = x \end{cases}$$

whose *transfer function* is given by:

$$\boxed{\frac{Y(s)}{U(s)} = \frac{1}{(s-a)}}$$

The only *pole* of this *system* is located in

$$s = a$$

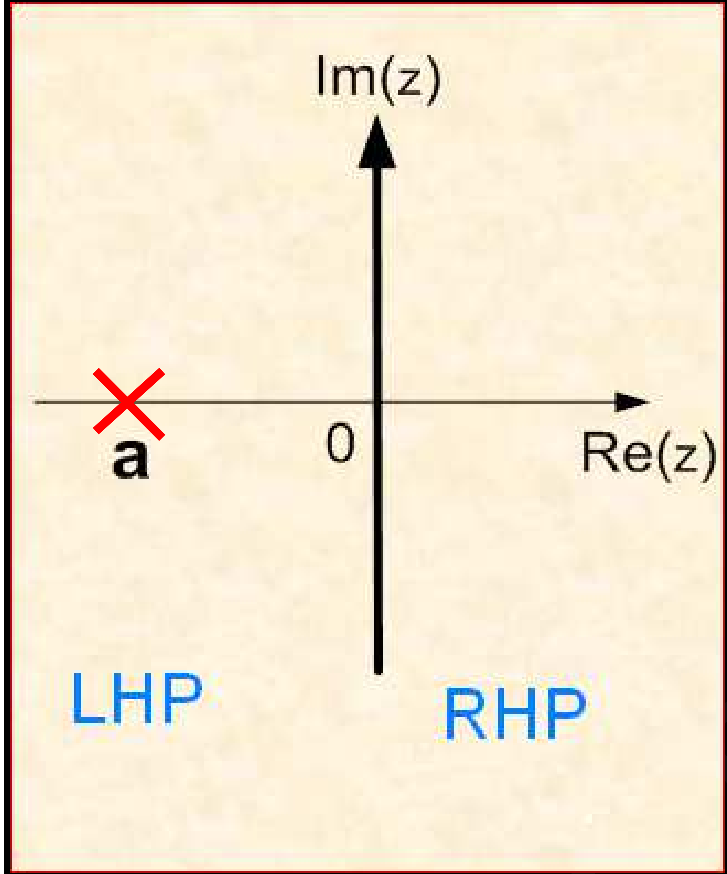
which can be in the LHP (*left half plane*), , on the *imaginary axis* or in the RHP (*right half plane*), depending on the value of a

($a < 0$, $a = 0$ or $a > 0$, respectively)

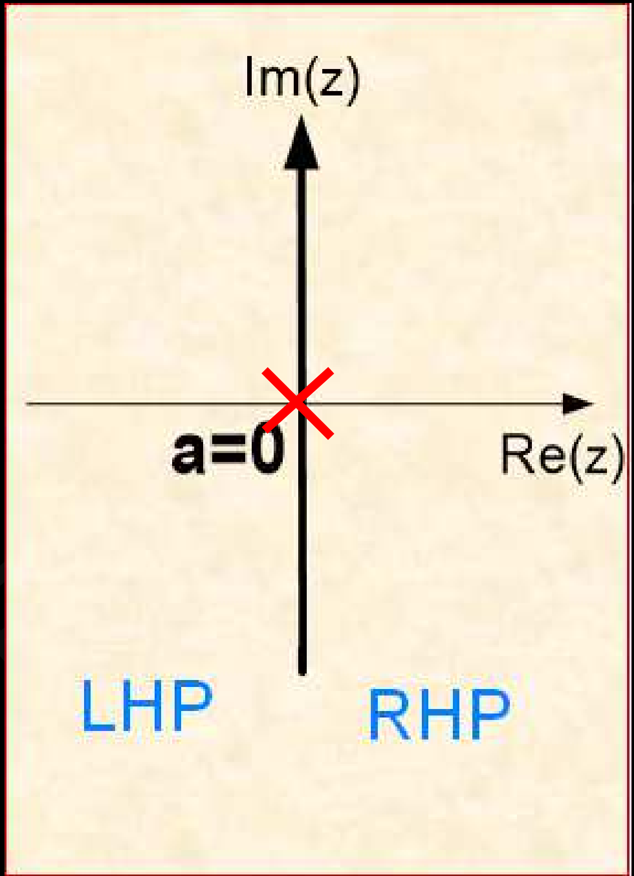
Stability

Example 1 (continued):

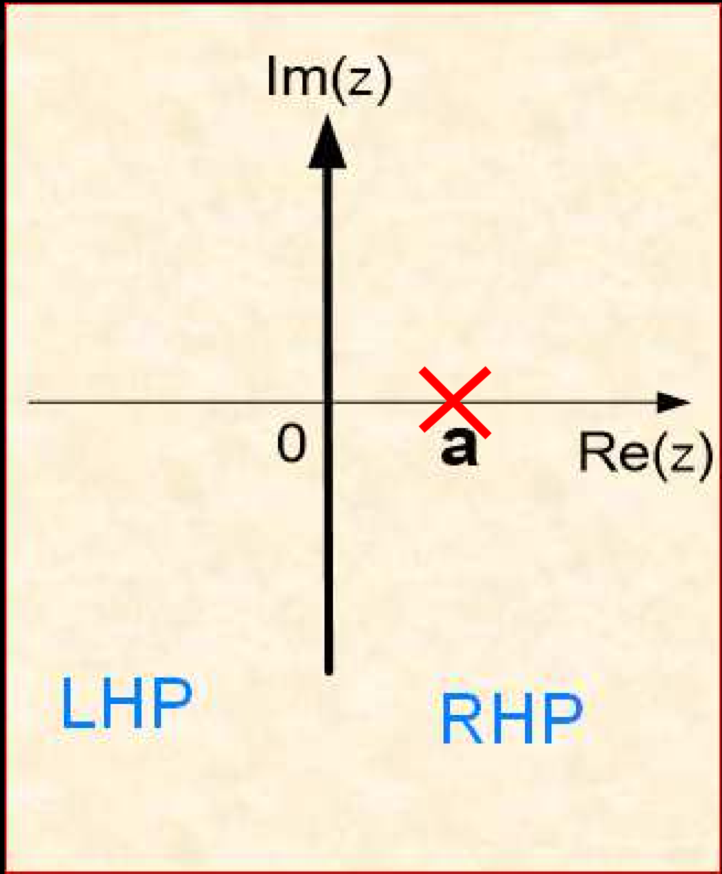
thus,



$a < 0$



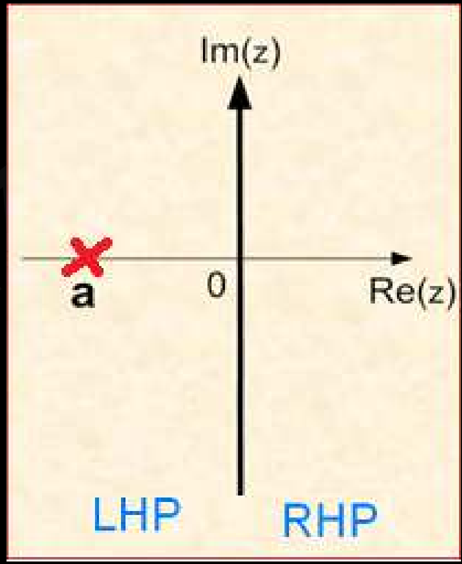
$a = 0$

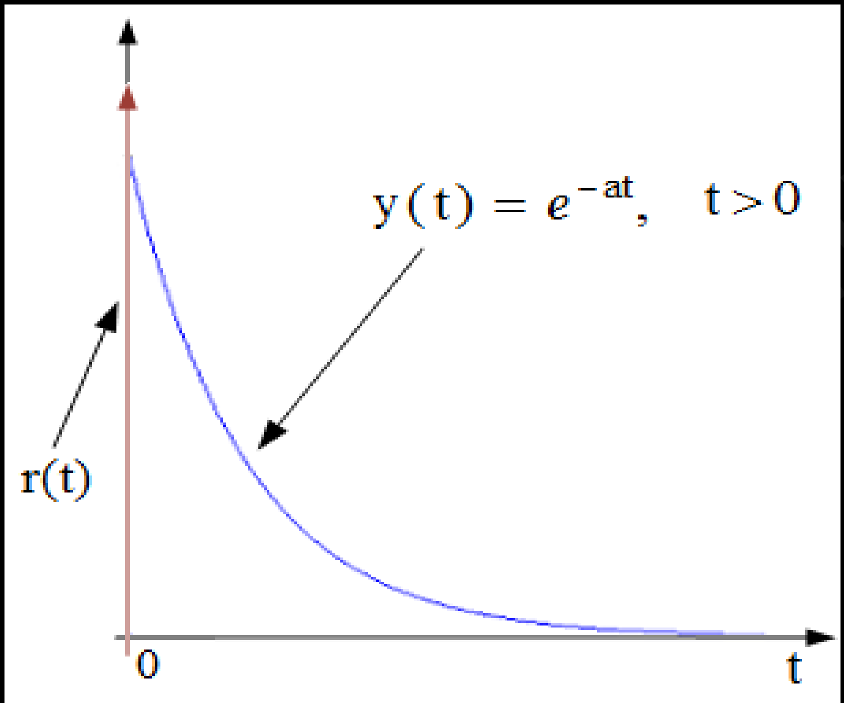


$a > 0$

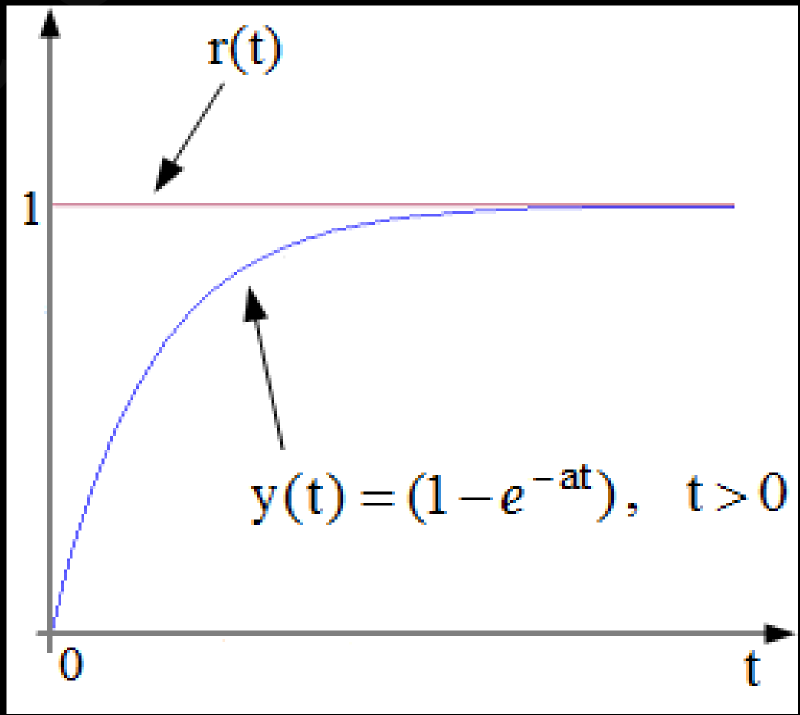
Example 1 (continued):

Case $a < 0$, *system* is *stable*



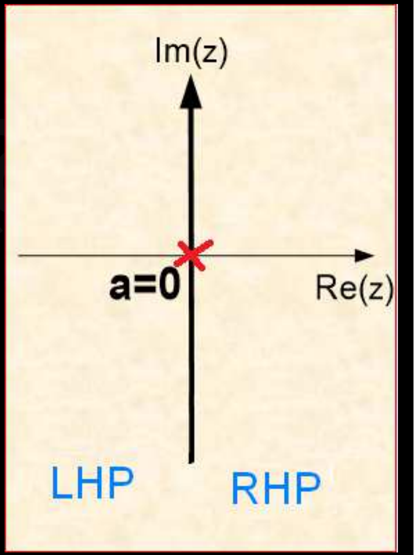


output to the *unit impulse*

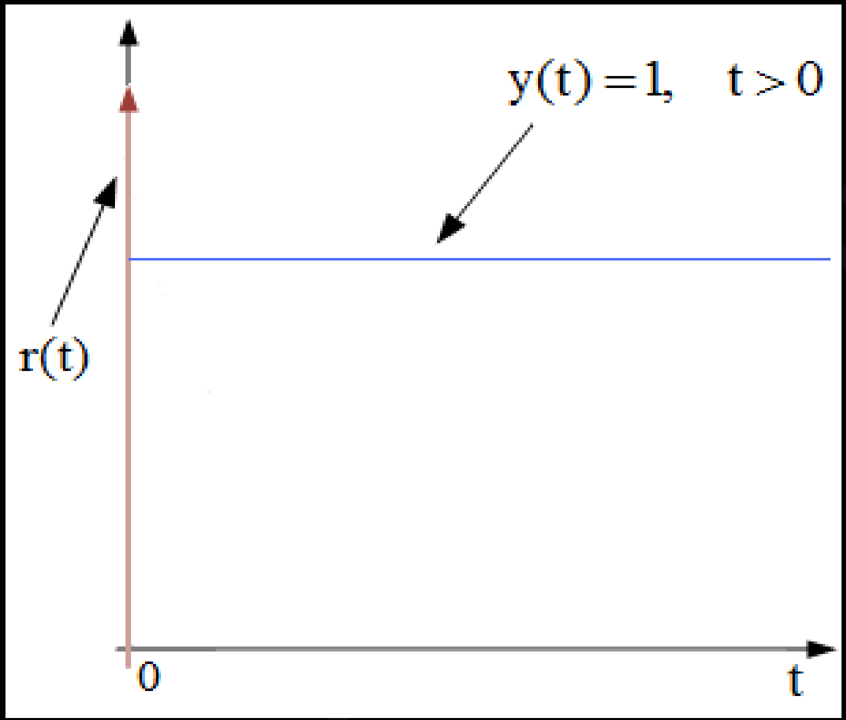


output to the *unit step*

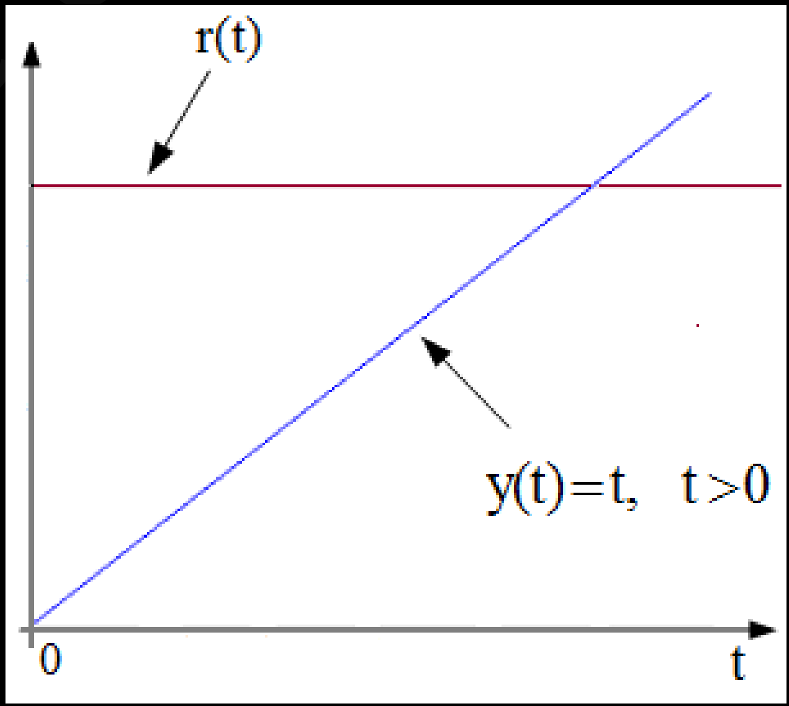
Example 1 (continued):



Case $\underline{a = 0}$, *system* is neither *stable* nor *unstable*



output to the *unit impulse*

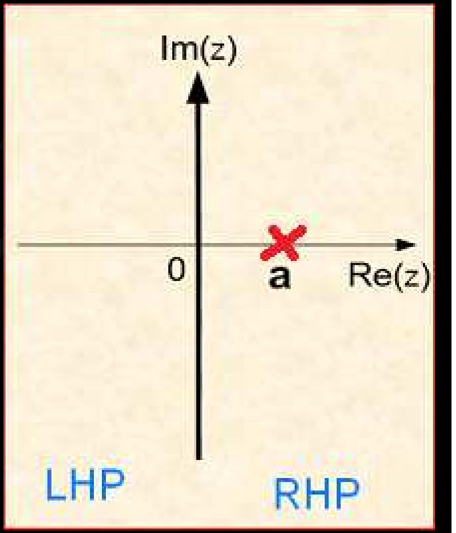


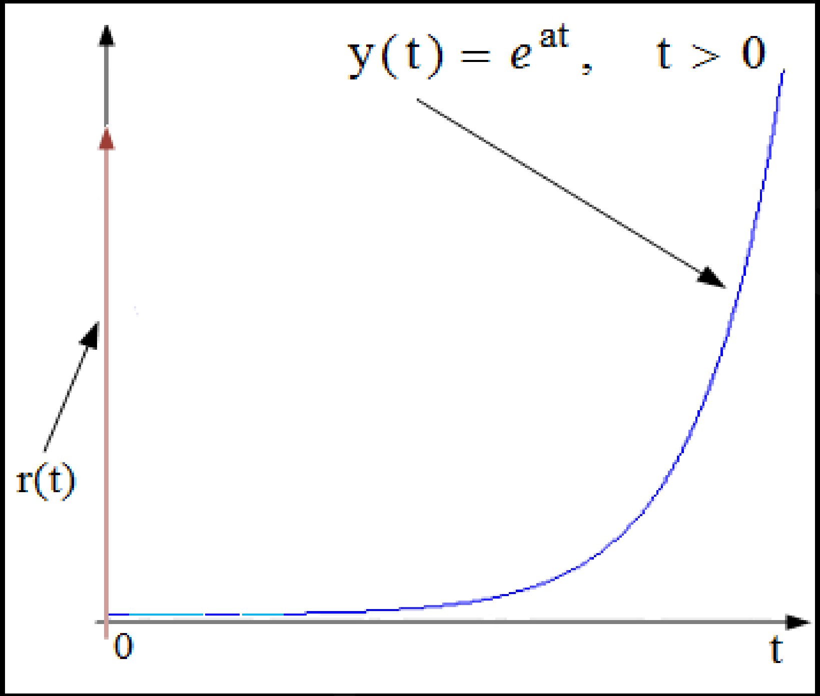
output to the *unit step*

Stability

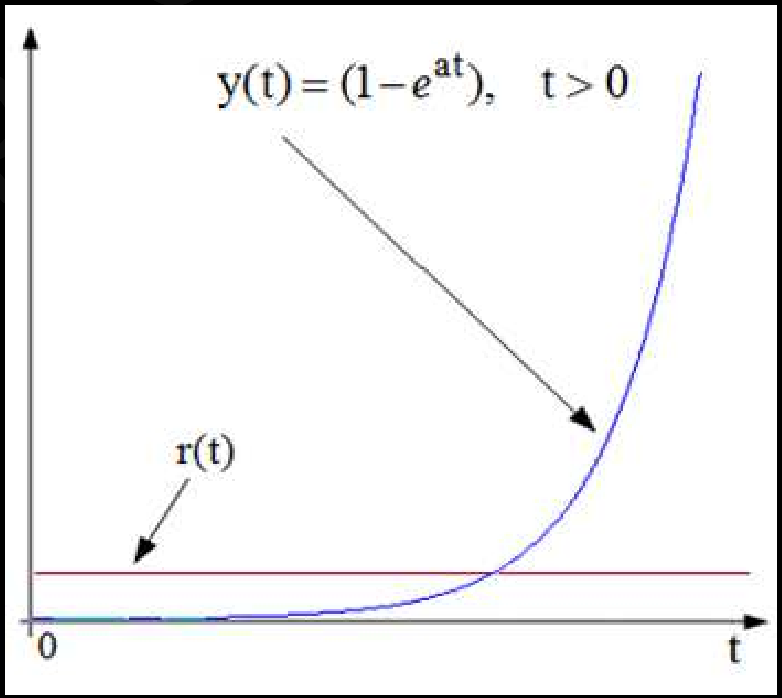
Example 1 (continued):

Case $a > 0$, *system* is *unstable*





output to the *unit impulse*

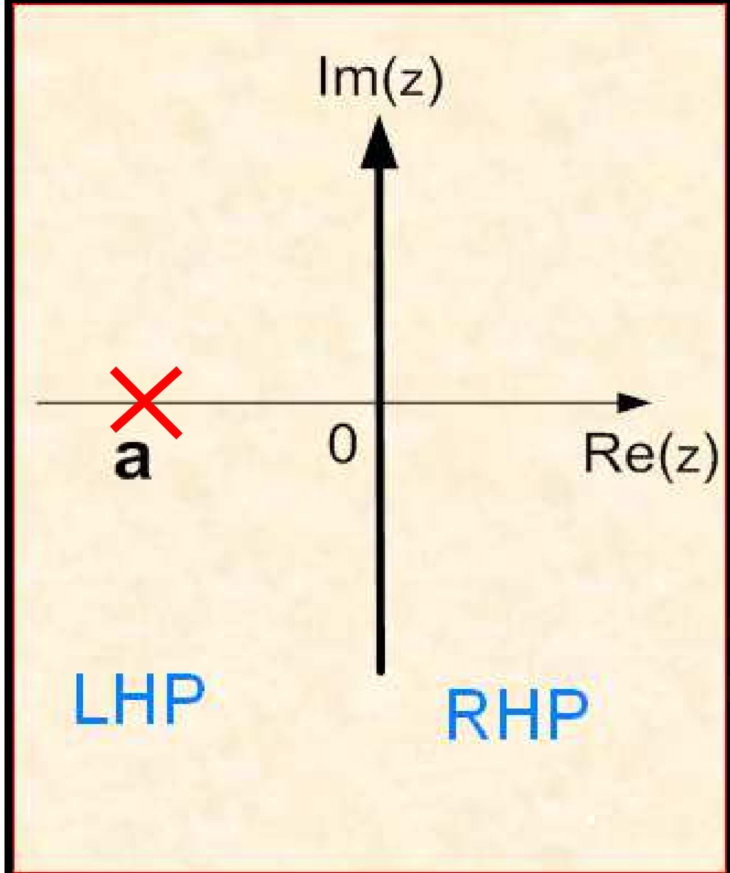


output to the *unit step*

Example 1 (continued):

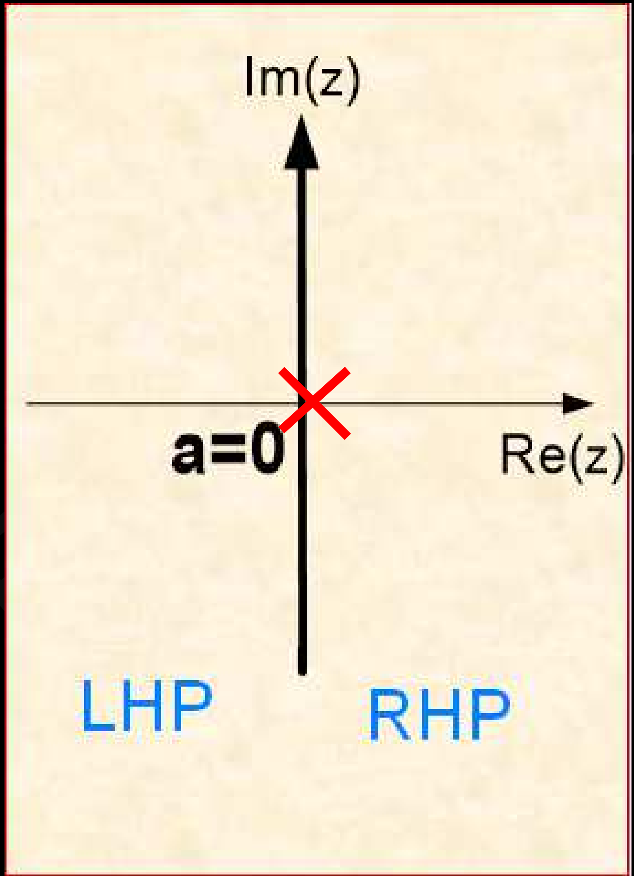
summarizing,

$a < 0$



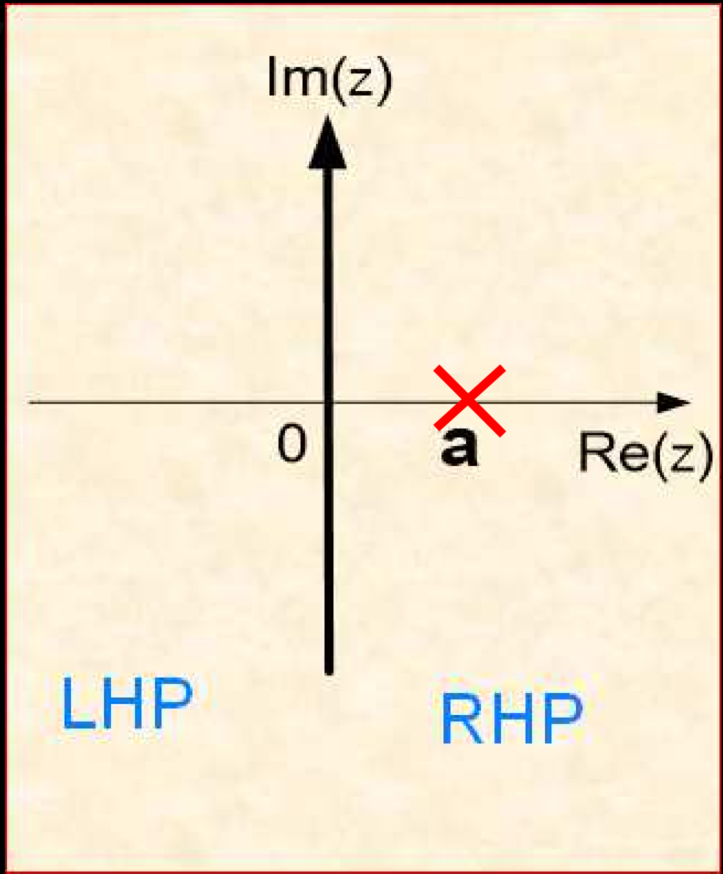
stable

$a = 0$



indifferent

$a > 0$



unstable

Stability

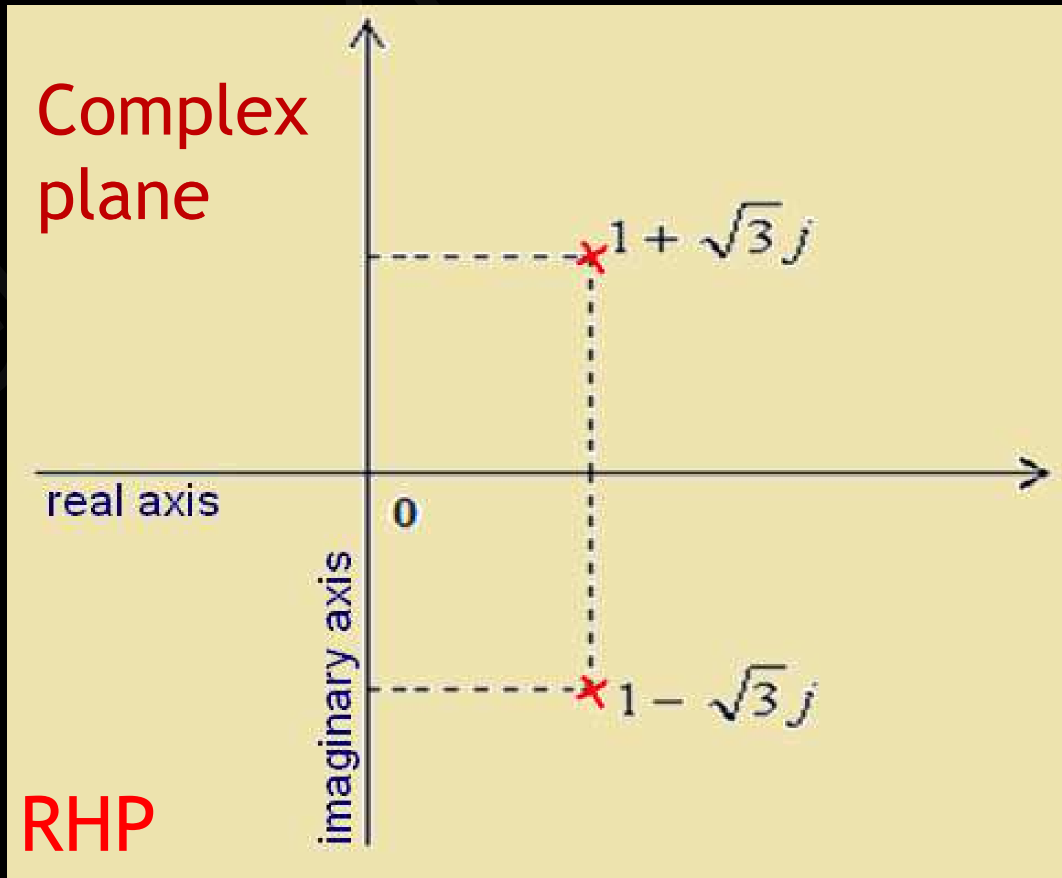
Example 2:

Consider the 2nd order *system* with transfer function given by

$$\frac{Y(s)}{U(s)} = \frac{4}{s^2 - 2s + 4}$$

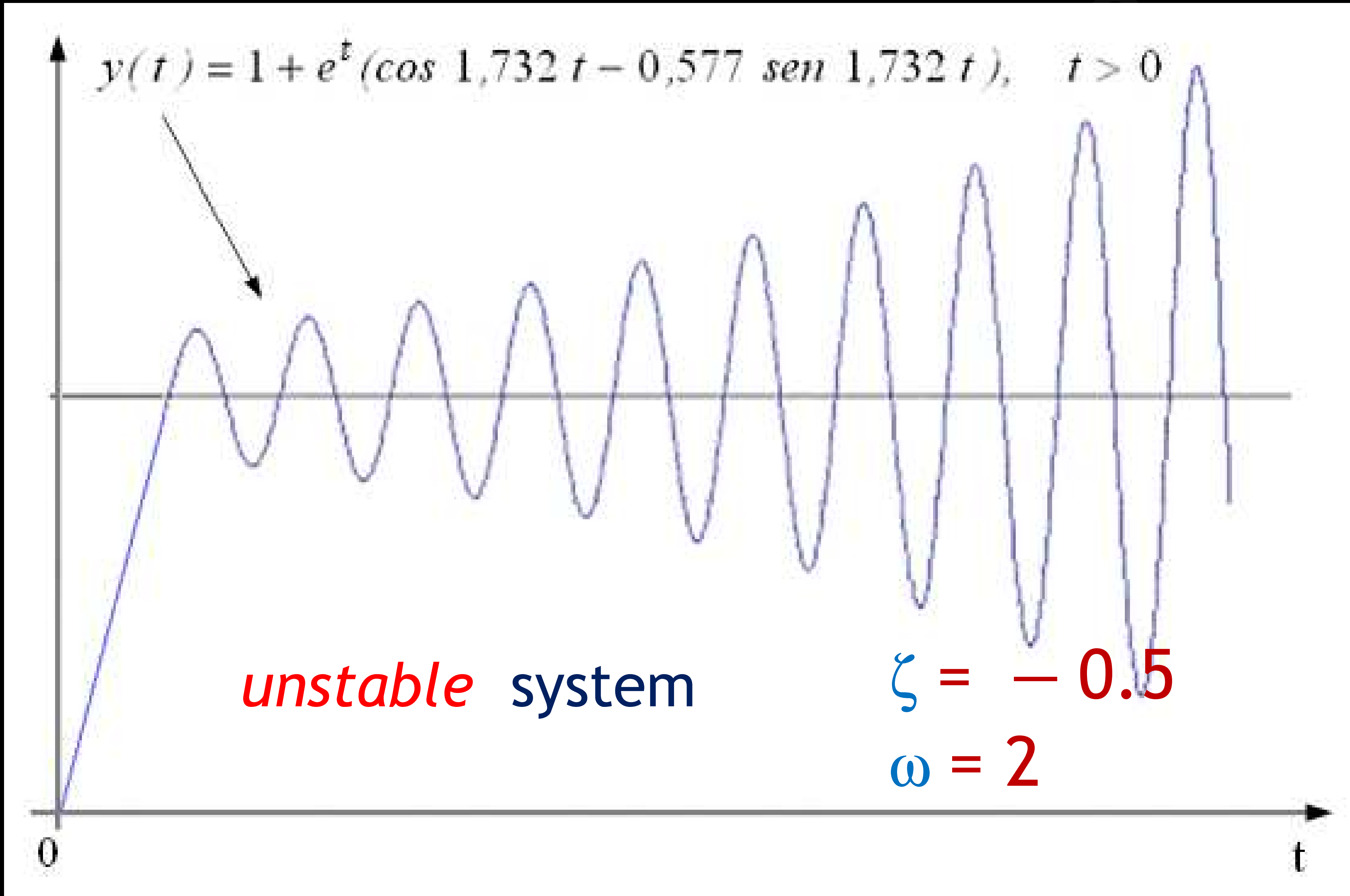
this *system* has got a pair of *complex poles* with positive real parts

$$s = 1 \pm j \cdot 1.732$$



poles in the RHP

Example 2 (continued):



output to the *unit step input*

Stability

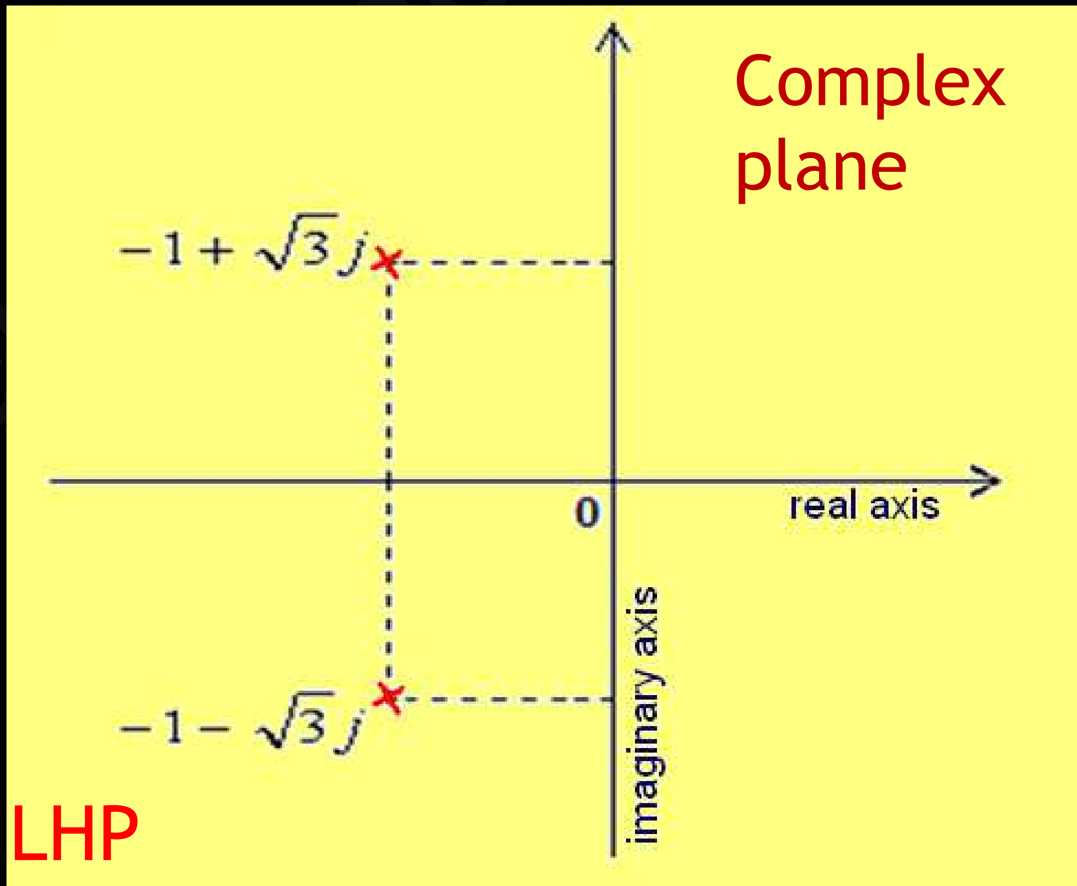
Example 3:

Consider the 2nd order *system* with transfer function given by

$$\frac{Y(s)}{U(s)} = \frac{4}{s^2 + 2s + 4}$$

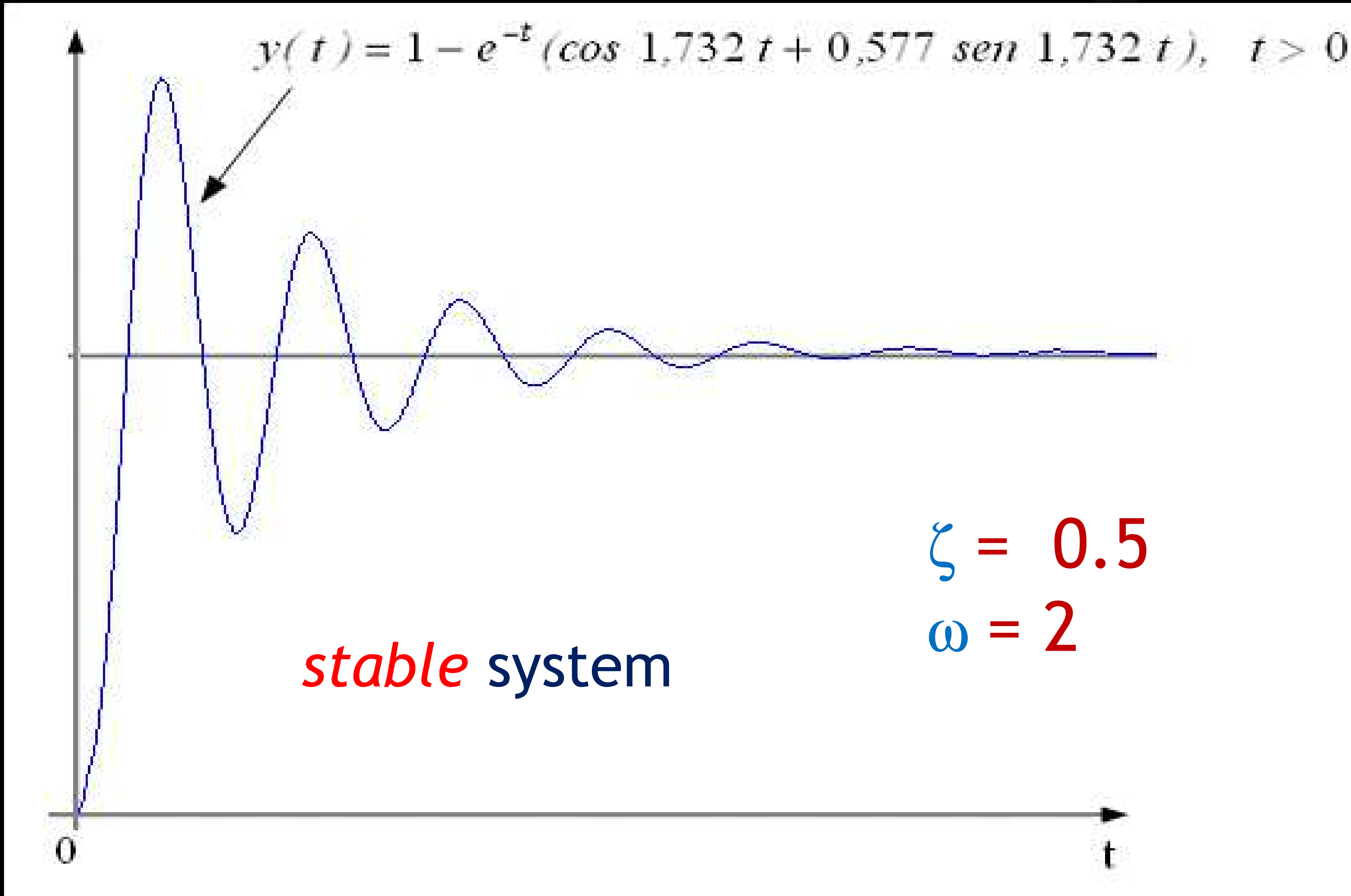
this *system* has got a pair of *complex poles* with negative real parts

$$s = -1 \pm j \cdot 1.732$$



poles in the LHP

Example 3 (continued):



output to the *unit step input*

Stability

Summarizing: the *stability* of *systems* depend only on the location of its *poles*, which have all to be in the LHP in order to the *system* to be stable

Thus, the *stability* do not depend of the whole *transfer function*, but only on its denominator, the

characteristic polynomial $p(s)$ of the system

that gives us the *poles* of the *system*

With respect to the *state equation*

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

the *stability* does not depend on B, C or D, but only on the matrix A that gives us the *characteristic polynomial* and the *poles* of the system

Routh-Hurwitz stability criterion

Routh, in the XIX century yet, in a time that there was no *electricity*, *calculating machine* or *computer*, create a mathematical method in order to determine the *number of poles* of a *system* located in the RHP without having to calculate the *poles* themselves

This made easier the determination of the *stability* of a *system*

(Canadian, 1831-1907)

Hurwitz is another important mathematician in the field of *Control* and *Dynamic systems*

Stability

Hurwitz derived in 1895 what is called today the

"Routh-Hurwitz stability criterion"

for determining whether a *system* is *stable*

It is said that Hurwitz did it independently of Routh who had derived it earlier by a different method

Adolf Hurwitz

(German, 1859-1919)

The Routh-Hurwitz criterion is constructed from the *characteristic polynomial* of the *system*

$$p(s) = a_o s^n + a_1 s^{n-1} + \cdots + a_{n-2} s^2 + a_{n-1} s + a_n \qquad (a_o > 0)$$

Stability

from $p(s)$ the Routh-Hurwitz table is built:

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}				
s^{n-3}				
:				
:				
s^2				
s^1				
s^o				

Routh-Hurwitz table, ***initially fulfilled with the coefficients of the characteristic polynomial*** $p(s)$

Routh-Hurwitz table, *initial fulfilling*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}				
s^{n-3}				
$\vdots$				
$\vdots$				
s^2				
s^1				
s^o				

If the *characteristic polynomial* $p(s)$ is odd, the last element stays in the 2nd line

Routh-Hurwitz table, *initial fulfilling*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	0
s^{n-2}				
s^{n-3}				
$\vdots$				
$\vdots$				
s^2				
s^1				
s^o				

If the *characteristic polynomial* $p(s)$ is even, the last element stays in the 1st line, and a '0' (zero) is put underneath in the 2nd line

Routh-Hurwitz table, *initial fulfilling*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}				
s^{n-3}				
$\vdots$				
$\vdots$				
s^2				
s^1				
s^o				

After the initial fulfilling the *remaining lines* are calculated as will be indicated a further here

Routh-Hurwitz table, after *completed*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^2	d_1	d_2		
s^1	e_1			
s^o	f_1			

After being completed the Routh-Hurwitz table will have (n+1) lines

Stability

Routh-Hurwitz table, after *completed*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^2	d_1	d_2		
s^1	e_1			
s^o	f_1			

Lines gets shorter and shorter up to the 2 *last lines* that have only one element each

Routh-Hurwitz table, *pivot column*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$			
s^2	d_1			
s^1	e_1			
s^o	f_1			

The first column (where are the coefficients a_o and a_1) is called '*pivot column*'

Routh-Hurwitz table, *pivot column*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
:	:	:	:	:
:	:			
s^2	d_1			
s^1	e_1			
s^o	f_1			

The following elements of the '*pivot column*' are: b_1, c_1, d_1, e_1, f_1, etc.

Routh-Hurwitz table, *pivot column*

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
$:$	$:$	$:$	$:$	$:$
$:$	$:$			
s^2	d_1			
s^1	e_1			
s^o	f_1			

The elements of the '*pivot column*' are called '*pivots*'

Stability

The elements b_1 , b_2 , b_3 , ... etc. of *third line*

$$b_1 = \frac{-1}{a_1} \cdot \det\begin{bmatrix} a_o & a_2 \\ a_1 & a_3 \end{bmatrix} = \frac{a_1 \cdot a_2 - a_0 \cdot a_3}{a_1}$$

$$b_2 = \frac{-1}{a_1} \cdot \det\begin{bmatrix} a_o & a_4 \\ a_1 & a_5 \end{bmatrix} = \frac{a_1 \cdot a_4 - a_0 \cdot a_5}{a_1}$$

$$b_3 = \frac{-1}{a_1} \cdot \det\begin{bmatrix} a_o & a_6 \\ a_1 & a_7 \end{bmatrix} = \frac{a_1 \cdot a_6 - a_0 \cdot a_7}{a_1}$$

Note that in the calculations of the elements of this line there is a *division* by a_1 , the *pivot* element of the *previous line*

Stability

The elements c_1 , c_2 , c_3 , … etc. of *third line*

$$c_1 = \frac{-1}{b_1} \cdot \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix} = \frac{b_1 \cdot a_3 - a_1 \cdot b_2}{b_1}$$

$$c_2 = \frac{-1}{b_1} \cdot \det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix} = \frac{b_1 \cdot a_5 - a_1 \cdot b_3}{b_1}$$

$$c_3 = \frac{-1}{b_1} \cdot \det \begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix} = \frac{b_1 \cdot a_7 - a_1 \cdot b_4}{b_1}$$

Note that in the calculations of the elements of this line there is a *division* by a_1, the *pivot* element of the *previous line*

In the same way we can calculate the *remaining elements* of all lines and columns and the Routh-Hurwitz table is completed

Note that, as already mentioned earlier, the lines of the Routh-Hurwitz table gets *shorter and shorter* as there are less elements to be computed in the last positions in every line

Observation: Routh-Hurwitz table was built supposing that the coefficient a_o of the *characteristic polynomial* $p(s)$ is *positive*,

$$a_o > 0$$

If the *coefficient* a_o is *negative*,

$$a_o < 0$$

then we can redefine $p(s)$ with all its *coefficients* having the signs changed (i.e., multiplied by -1)

Therefore we get $a_o > 0$

This happens since, as it is well known, the *roots of a polynomial* do not alter when all its coefficients are multiplied by -1 (*or by any other constant value* $\neq 0$).

Stability

To simplify the calculations, if we wish we can *multiply* (or *divide*) all elements of any line of the Routh-Hurwitz table by a *positive number*

This will not alter the result to be obtained from the Routh-Hurwitz table

The Routh-Hurwitz table allows us to find out how many *poles* are located in the RHP

The number of **sign changes in the pivot column** of Routh-Hurwitz table is the number of *poles* in the RHP

Routh-Hurwitz table

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
:	:	:	:	:
:	:			
s^2	d_1			
s^1	e_1			
s^o	f_1			

the number of ***sign changes*** in the ***pivot column*** is the number of ***poles in the LHP***.

The determination of the *number of poles* in the RHP can be very useful but, nevertheless, it doesn't give us a diagnosis for stability of the *system* immediately.

That is because a *system* to be *stable* must have all its *poles* in the LHP and therefore, even if the *number of sign changes* in the *pivot column* is zero, this only means that there will be 'zero *poles*' in the RHP, which does not guarantee stability yet, since it may have some *pole* in the *imaginary axis*

However, *poles* in the *imaginary axis* will reflect in zeros in the *pivot column*. These allows us to write the following result:

> A *system* has all its *poles* in the LHP if, and only if, all the coefficients of the *pivot column* of the Routh-Hurwitz table are *positives*

Stability

Routh-Hurwitz table

s^n	a_o	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	b_4
s^{n-3}	c_1	c_2	c_3	c_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$			
s^2	d_1			
s^1	e_1			
s^o	f_1			

The system has all its *poles* in the LHP whenever all its *pivot column* elements are *positives*

Example 4:

Consider now the *characteristic polynomial of* degree 4 given below

Setting up the Routh-Hurwitz table we get

$$p(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	–6		
s^0	5		

2 sign changes (*of elements in the pivot column*)

This *characteristic polynomial* has 2 *poles* in the RHP

Thus, the *system* is not stable

Example 4 (continued):

Here we could, for example, to simplify the calculation, divide the 2nd line (i.e., line s^3) by 2

This do not alter the final result

$$p(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

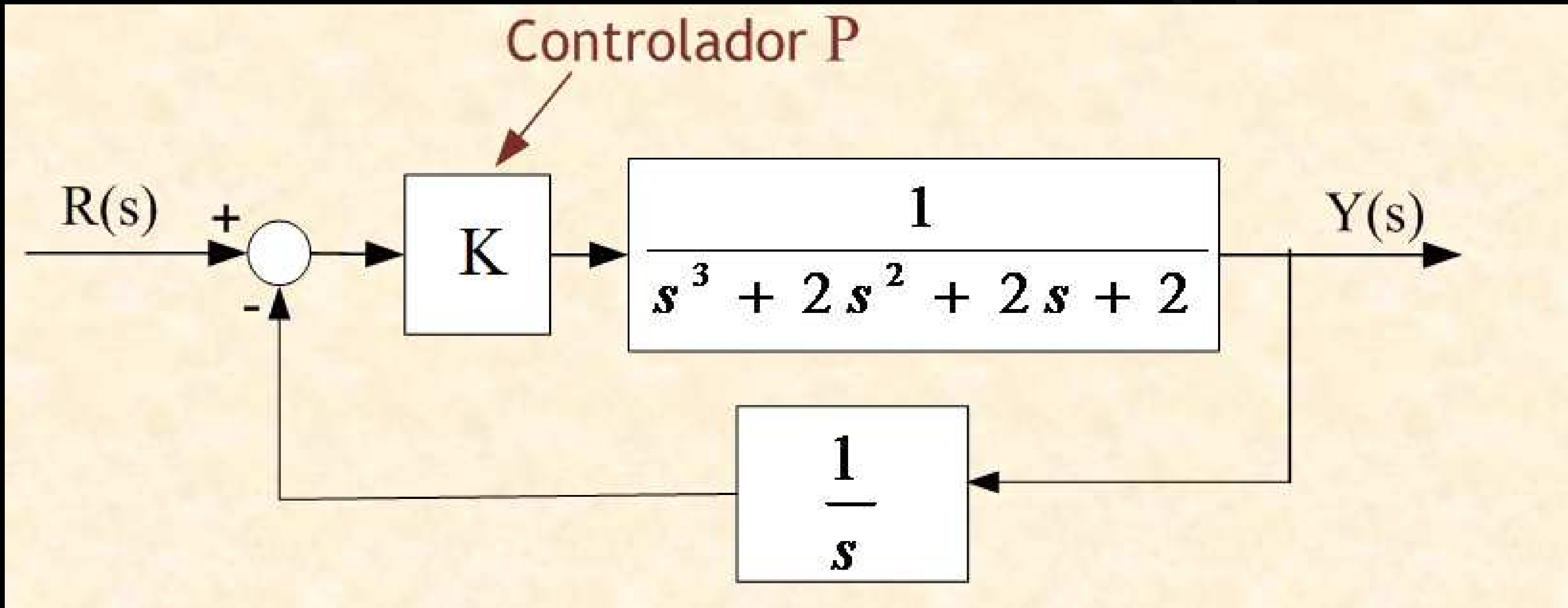
s^4	1	3	5
s^3	1	2	0
s^2	1	5	
s^1	–3		
s^0	5		

← 2nd line ÷ 2

and we get to the same conclusion: 2 sign changes (*of elements of the pivot column*)

Example 5:

Find the values of K for which the *closed loop system* below is stable



Computing the *closed loop transfer function* (CLTF), we get

$$\frac{Y(s)}{R(s)} = \frac{Ks}{s^4 + 2s^3 + 2s^2 + 2s + K}$$

Example 5 (continued):

So, the *characteristic polynomial* of the *closed loop system* is given below

The Routh-Hurwitz table for this *polynomial* is:

$$p(s) = s^4 + 2s^3 + 2s^2 + 2s + K$$

s^4	1	2	K
s^3	2	2	0
s^2	1	K	
s^1	$(2-2K) > 0$		
s^0	$K > 0$		

$$(2-2K) > 0 \Rightarrow K < 1$$

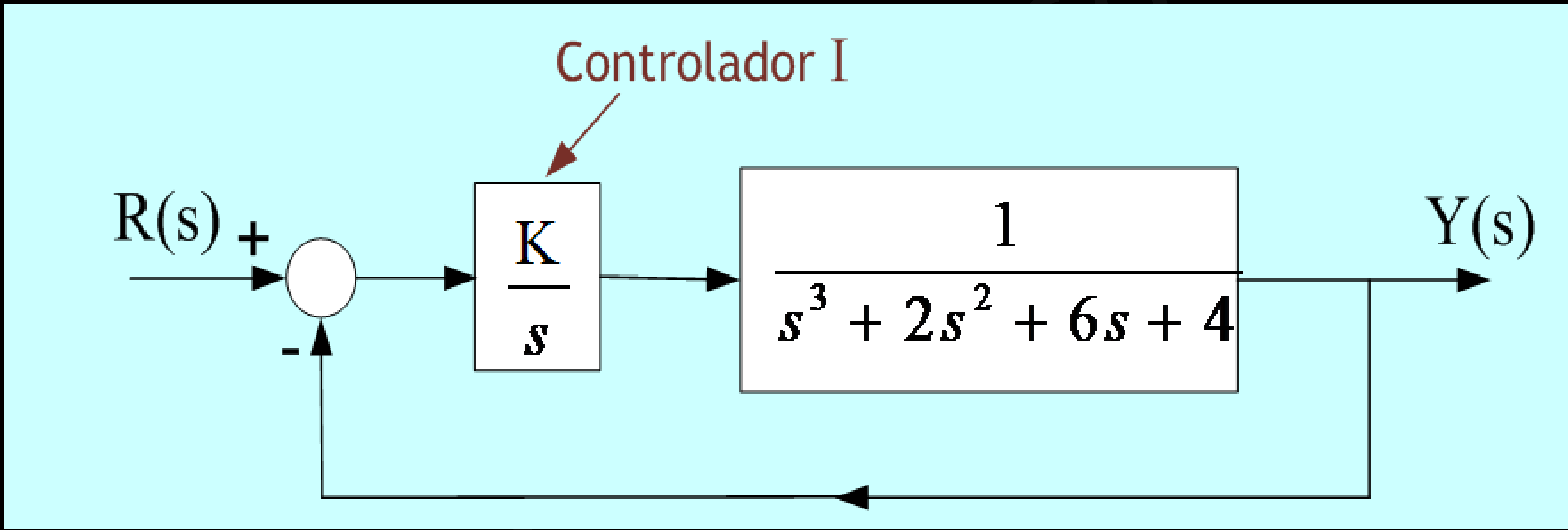
$$\left.\begin{array}{l} K < 1 \\ K > 0 \end{array}\right\} \Rightarrow \boxed{0 < K < 1}$$

system is stable for $0 < K < 1$

Stability

Example 6:

Find the values of K for which the *closed loop system* below is stable



It is easy to verify that the *characteristic polynomial* of this *system* is:

$$p(s) = s^4 + 2s^3 + 6s^2 + 4s + K$$

Example 6 (continued):

thus, the Routh-Hurwitz table for this *polynomial* is:

$$p(s) = s^4 + 2s^3 + 6s^2 + 4s + K$$

s^4	1	6	K
s^3	2	4	0
s^2	4	K	
s^1	$(4-K/2) > 0$		
s^0	$K > 0$		

$(4-K/2) > 0 \Rightarrow K < 8$

$\left.\begin{matrix} K < 8 \\ K > 0 \end{matrix}\right\} \Rightarrow$ *system is* stable for $\boxed{0 < K < 8}$

Example 7:

Consider now the *characteristic polynomial* p(s) of 3rd degree below

The Routh-Hurwitz table is also given below

$$p(s) = s^3 + 3s^2 + s + 3$$

s^3	1	1
s^2	3	3
s^1	$0 \cong \varepsilon > 0$	
s^0	3	

so, all the elements of the *pivot column* are positives, that is

0 (*zero*) roots in the RHP

Actually, this 'zero' in the *pivot column* indicates a symmetry of 2 *poles* and they can only be located in the *imaginary axis*

Example 7:

Consider now the *characteristic polynomial* p(s) of 3rd degree below

The Routh-Hurwitz table is also given below

$$p(s) = s^3 + 3s^2 + s + 3$$

s^3	1	1
s^2	3	3
s^1	$0 \cong \varepsilon > 0$	
s^0	3	

so, all the elements of the *pivot column* are positives, that is

0 (*zero*) roots in the RHP

Thus, the *system* is not unstable, but it does not have all its *poles* in the LHP either, that prevents stabilization

Example 8:

Consider now the *characteristic polynomial* p(s) of 5th degree below

$$p(s) = s^5 + 6s^4 + 13s^3 + 14s^2 + 6s$$

s^5	1	13	6
s^4	6	14	0
s^3	32/3	6	
s^2	340/32	0	
s^1	6		
s^0	0 ←		

so, there is no sign change in the *pivot column*, and that means 0 (*zero*) roots in RHP

0 (*zero*) in *pivot column*

That zero in the *pivot column* is in the last position.
(It indicates symmetry of 1 *pole* with respect to the origin, that is, *one pole* = zero)

Example 8:

Consider now the *characteristic polynomial* p(s) of 5th degree below

$$p(s) = s^5 + 6s^4 + 13s^3 + 14s^2 + 6s$$

s^5	1	13	6
s^4	6	14	0
s^3	32/3	6	
s^2	340/32	0	
s^1	6		
s^0	0		

$\longleftarrow$ 0 (*zero*) in *pivot column*

so, there is no sign change in the *pivot column*, and that means 0 (*zero*) roots in RHP

So, the *system* is <u>not</u> unstable, nevertheless only 4 of its 5 *poles* are in the LHP, since one is in the origin ($s = 0$)

Example 9:

Consider now the *characteristic polynomial below* of 5^{th} degree

$$p(s) = s^5 + 3s^4 + 12s^3 + 36s^2 - 29s - 87$$

s^5	1	12	–29
s^4	3	36	–87
s^3	0	0	0

$\rightarrow$ $\left\{ \begin{array}{l} 0 \text{ in } \textit{pivot column} \\ \text{line of zeroes} \end{array} \right.$

from line s^4 we get $q(s)$

$$q(s) = 3s^4 + 36s^2 - 87$$

and find the derivative $q'(s)$

$$q'(s) = \frac{dq}{ds} = 12s^3 + 72s$$

Example 9 (continued):

By eliminating the *line of zeroes*, we can complete the Routh-Hurwitz table

$$p(s) = s^5 + 3s^4 + 12s^3 + 36s^2 - 29s - 87$$

s^5	1	12	–29	
s^4	3	36	–87	
s^3	~~0~~	~~0~~	~~0~~	line of zeroes out
s^3	12	72	0	line q’(s) gets in
s^2	18	–87		
s^1	130			
s^0	–87			

1 sign change (*of elements of the pivot column*)

This *characteristic polynomial* has 1 pole in RHP

Thus, the *system* is not stable.

Stability

relative stability

Relative Stability

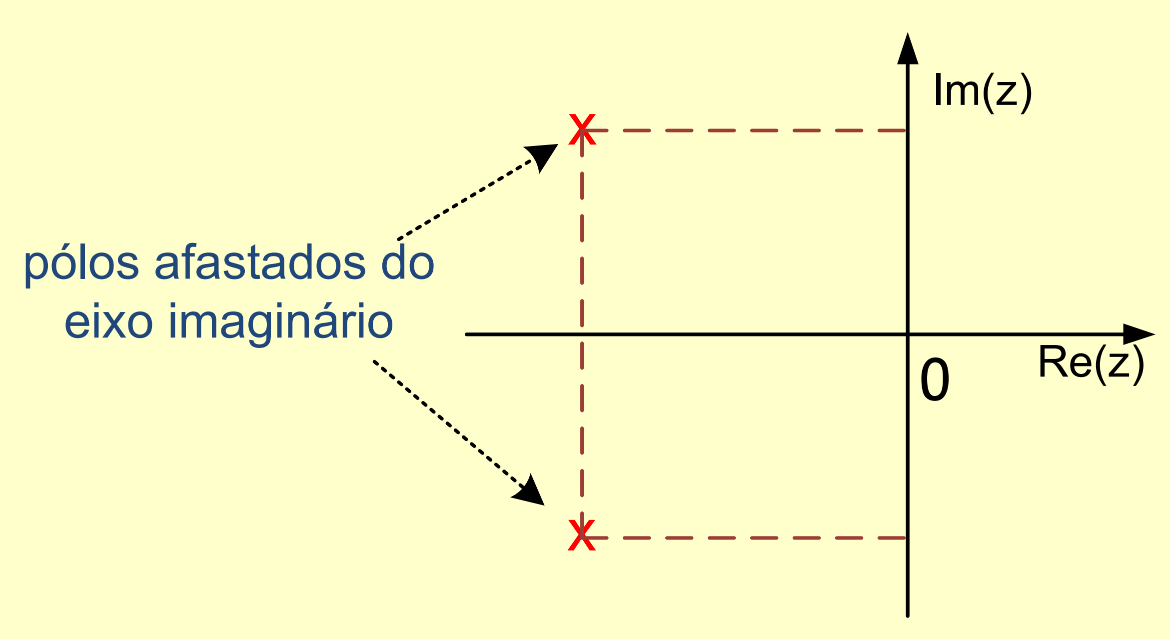
Sometimes it is desirable that the *poles* do not lie close to the imaginary axis $j\omega$ (*in order to avoid having slow input responses or with excessive oscillations*)

What makes the stabilization to be *quicker* or *slower* is the *localization of the poles* to be *further* or *closer* (*to the left*) to the *imaginary axis*

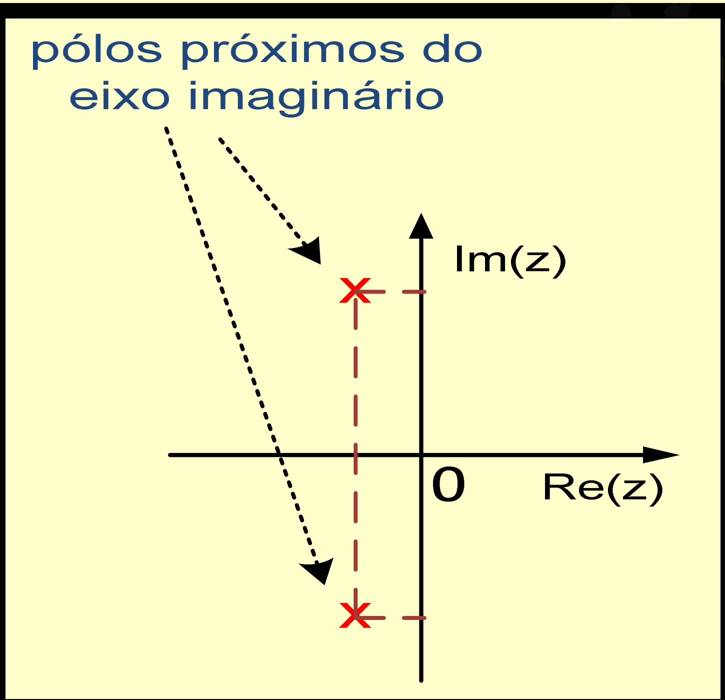
The *further* to the left are the *poles of the system*, the *quicker* it stabilizes.

Stability

Consider these 2 *systems*:



System A
with *poles* far from *imaginary axis*



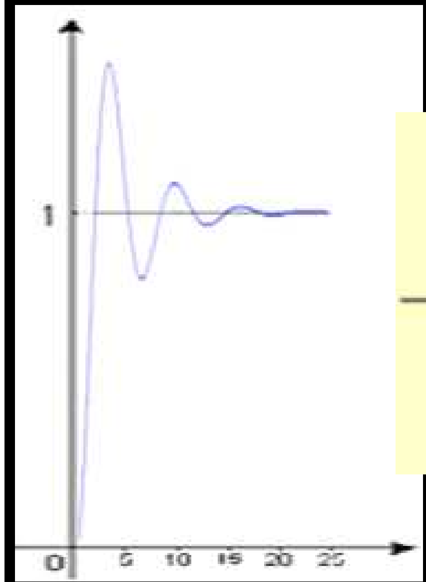
System B
with *poles* close to the *imaginary axis*

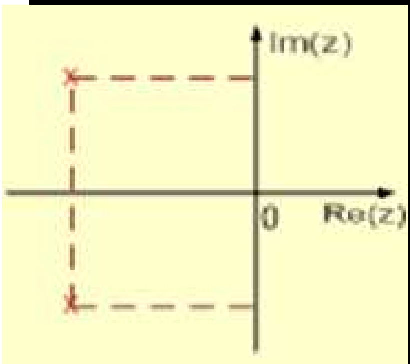
Stability

step response

System A

takes only 25 seconds to stabilize

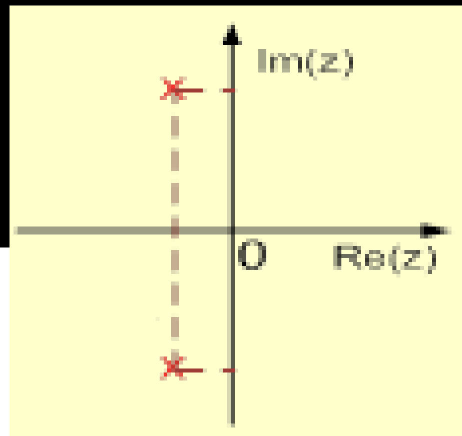


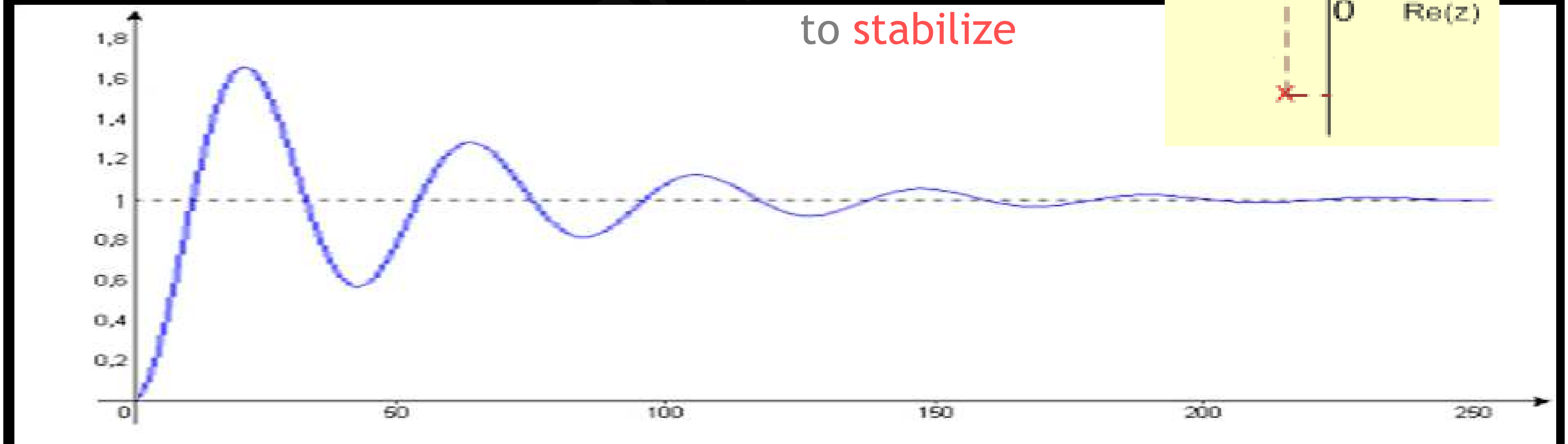


System A stabilizes quicker than system B since it has its *poles* further to the *imaginary axis*

System B

takes 250 seconds to stabilize





Stability

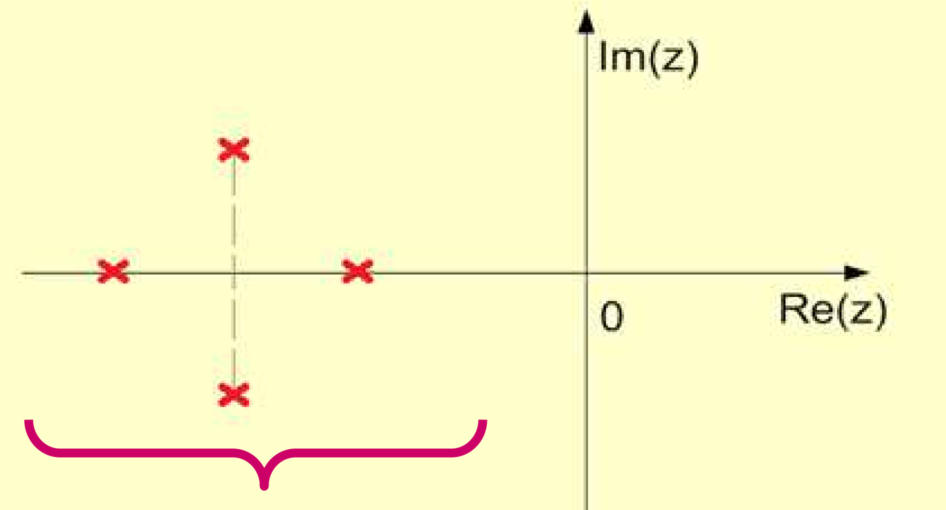
We have already seen that a *system* is stable or not if it has all its *poles* in the LHP or not, respectively

That is the concept of "*ABSOLUTE STABILITY*".

However, we have just seen too that: a stable *system* can be more stable than other

That depends on how further away from the *imaginary axis*, to the left, are located the *poles* of this *system*

That is the concept of "*relative stability*"

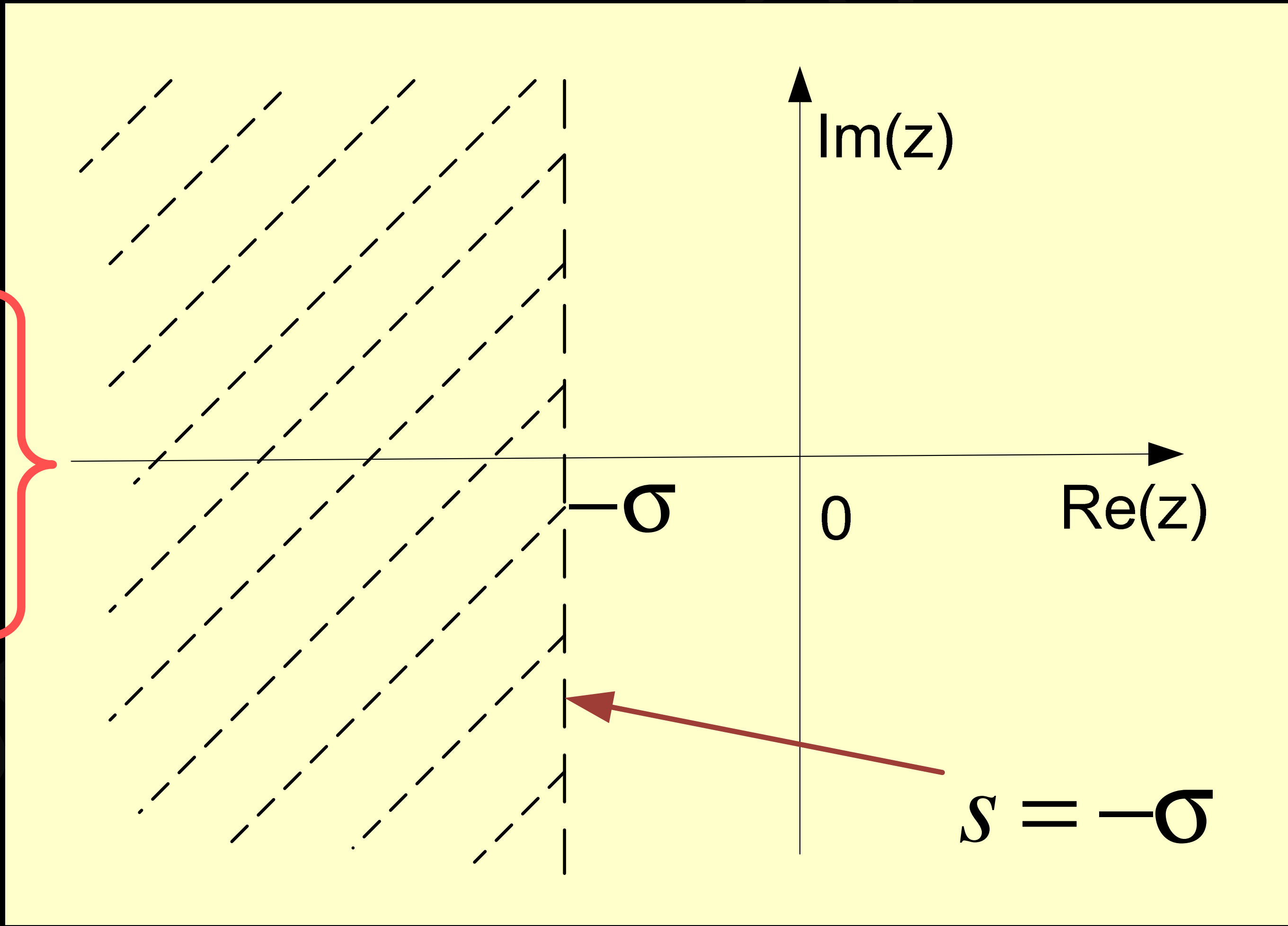


The location of the poles is what counts.

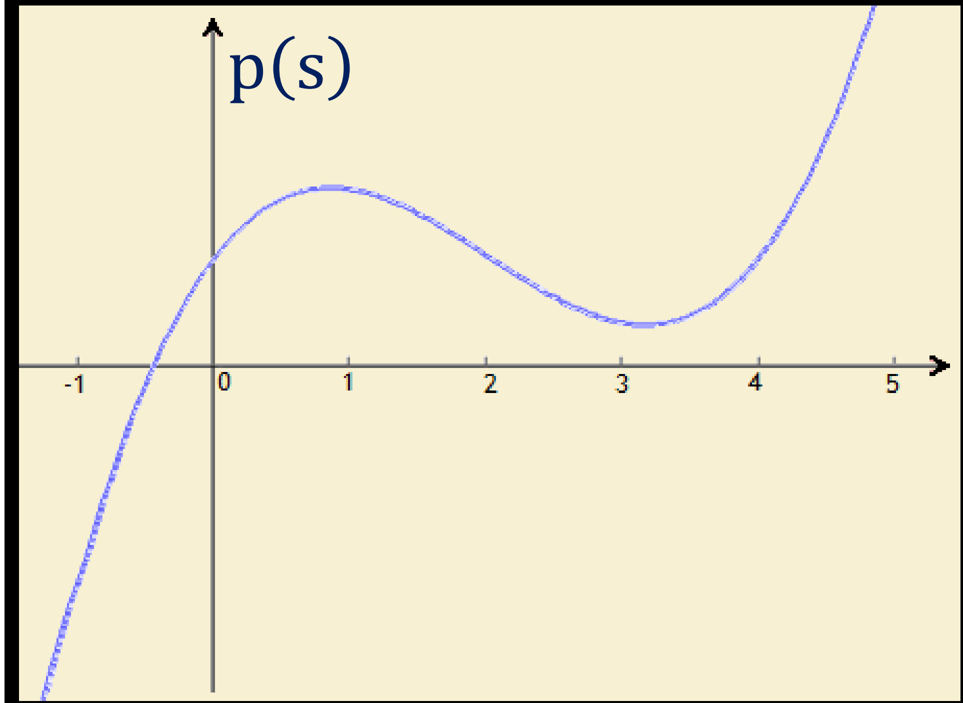
Stability

So, it is possible that we wish to verify if a *system* has its *poles* *lying* to the left of the line $s = -\sigma$ in the LHP, and not only in the LHP

Region of the LHP to the left of the line $s = -\sigma$



Stability



To obtain that we do a translation

$$s \mapsto (s - \sigma)$$

of the *characteristic polynomial* $p(s)$ so that we get

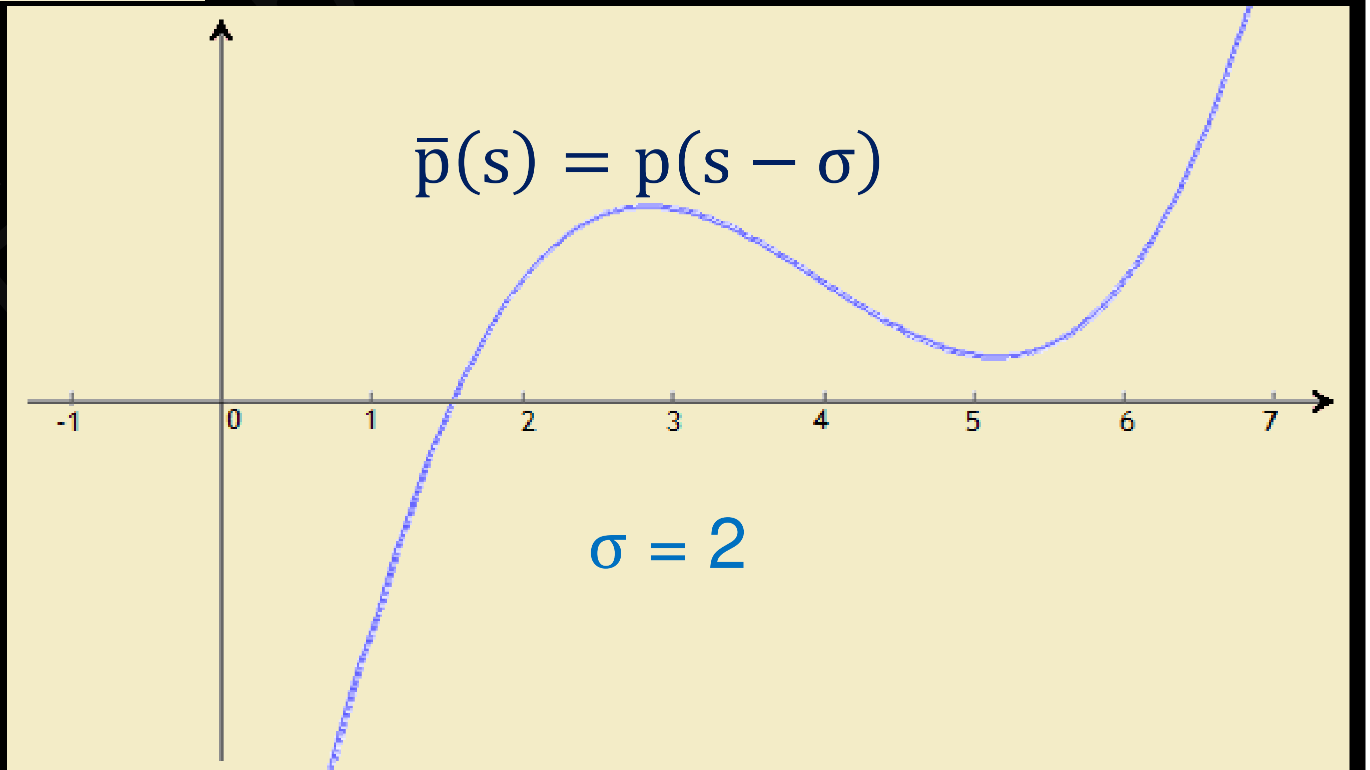
$$\bar{p}(s) = p(s - \sigma)$$

Which is the *polynomial* $p(s)$ shifted to σ to the right

In this example

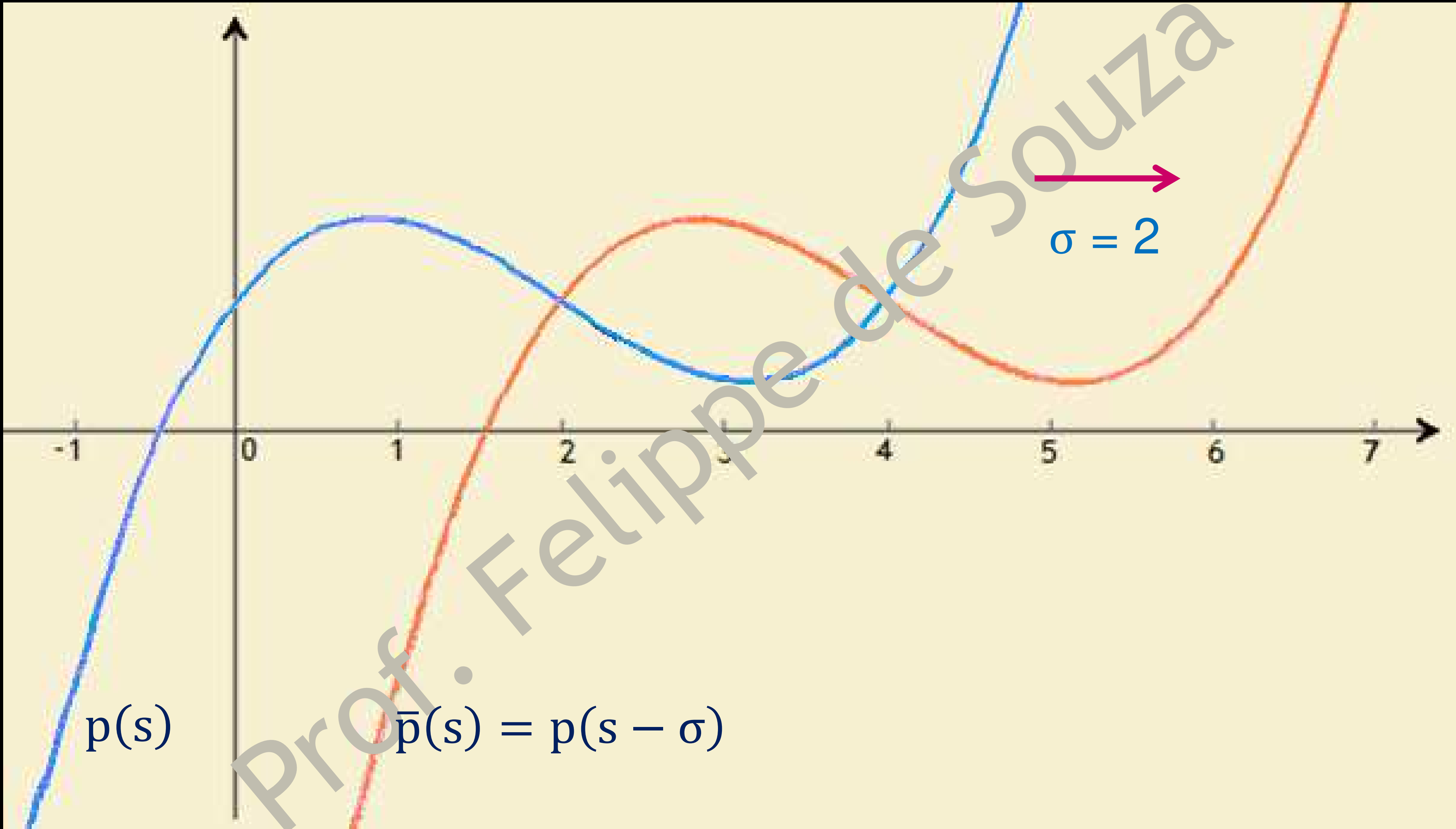
$$\sigma = 2$$

$$\bar{p}(s) = p(s - 2)$$



Stability

Translation of the polynomial to the right



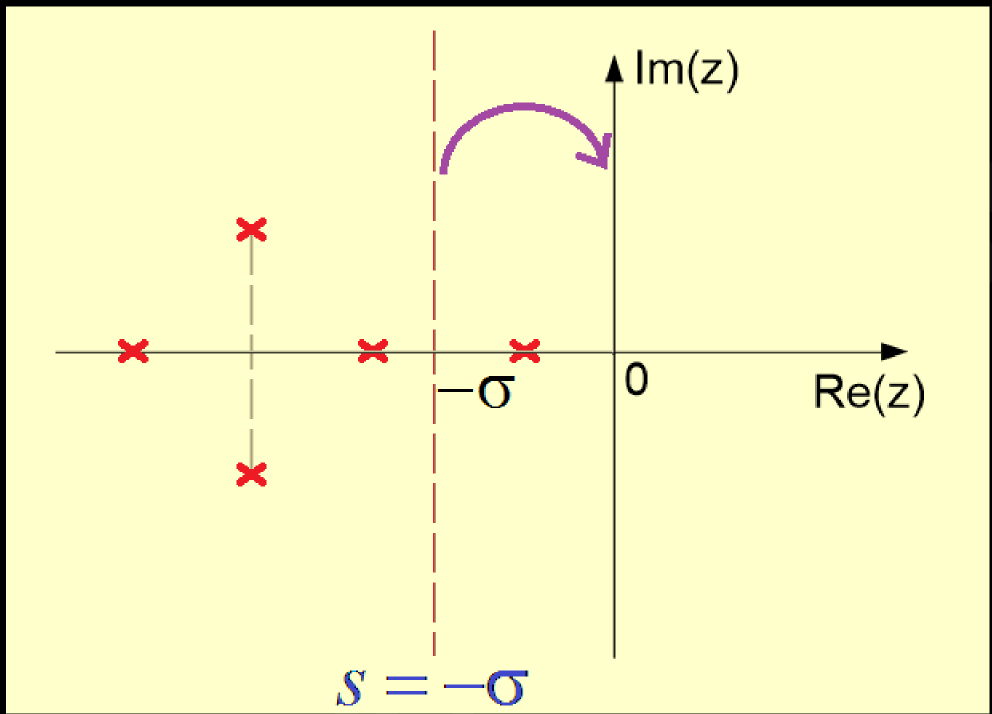
Stability

A translation of the polynomial to the right corresponds to

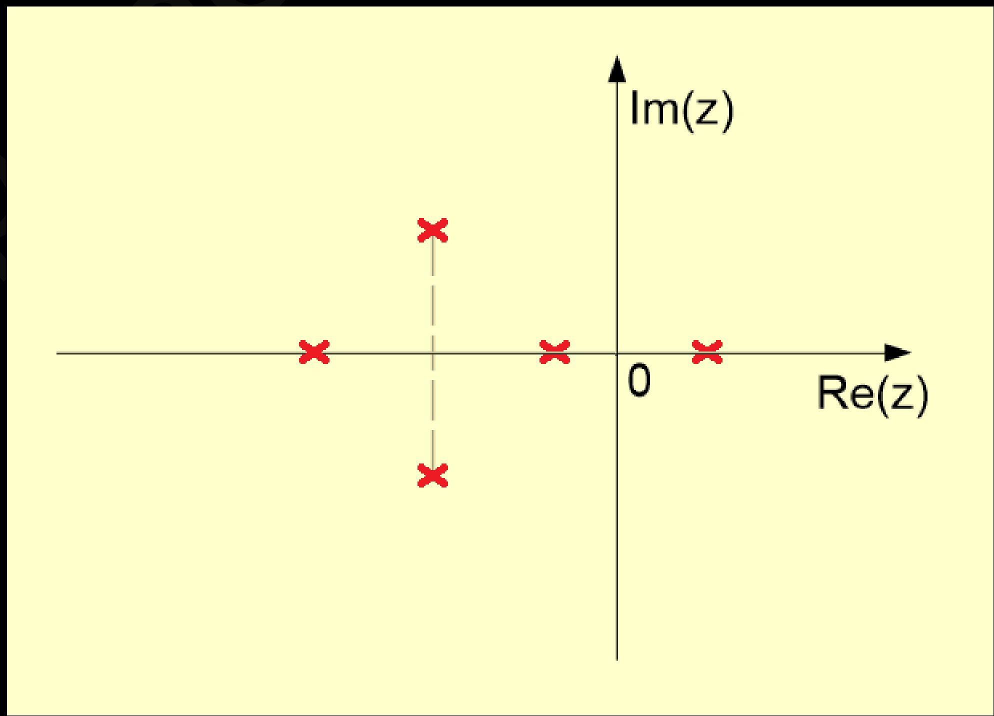
shift the *line* $s = -\sigma$ to the *right*

or, equivalently,

shift the *imaginary axis* to the *left*



roots of $p(s)$

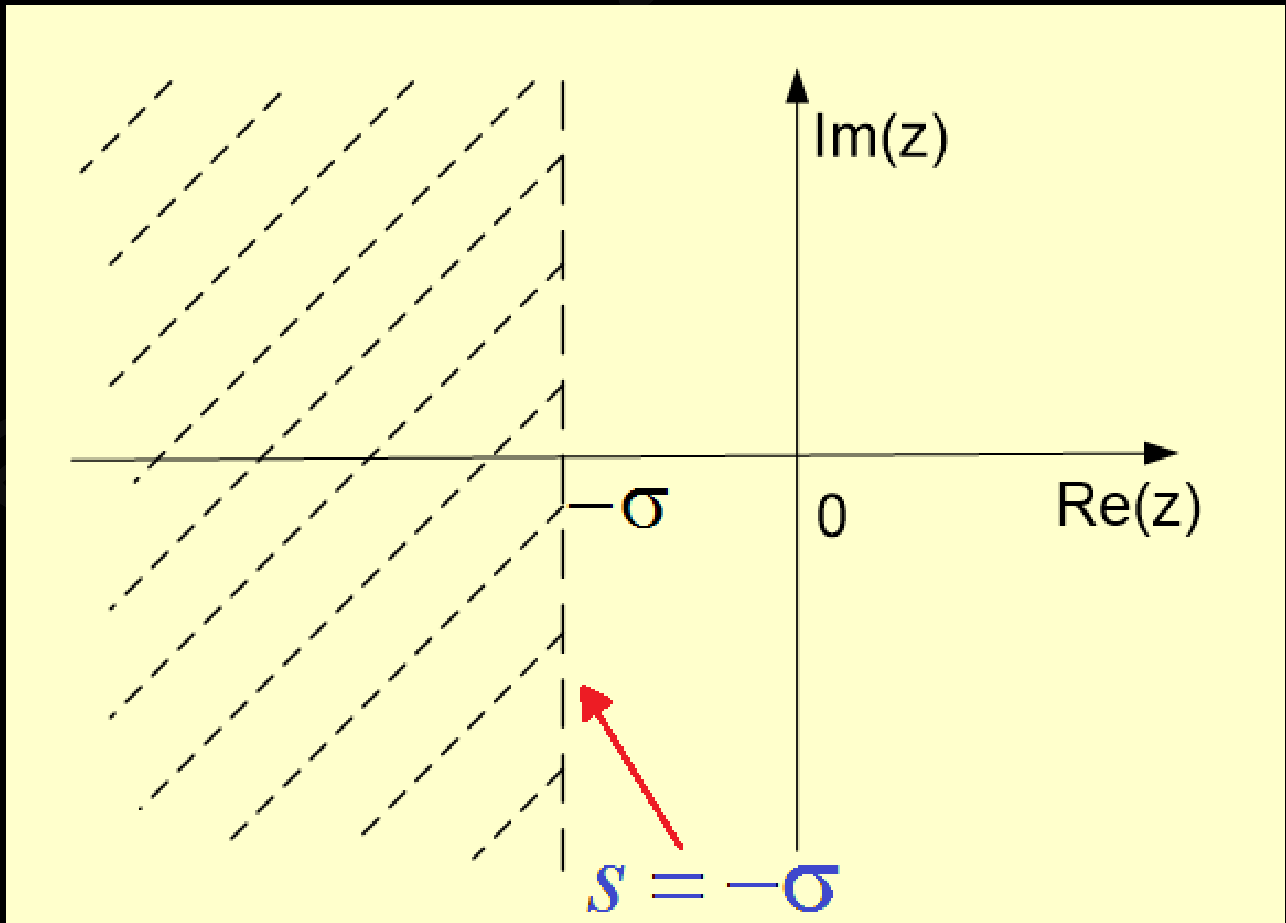


roots of $\bar{p}(s) = p(s - \sigma)$

Stability

Applying the "Routh-Hurwitz stability criterion" to the translated polynomial $\bar{p}(s)$, we have that o number of *sign changes* in the *pivot column* is equal to the *number of roots* of $p(s)$ located to the *right* of the line $s = -\sigma$

That is, through $\bar{p}(s)$ we can extract conclusions for $p(s)$



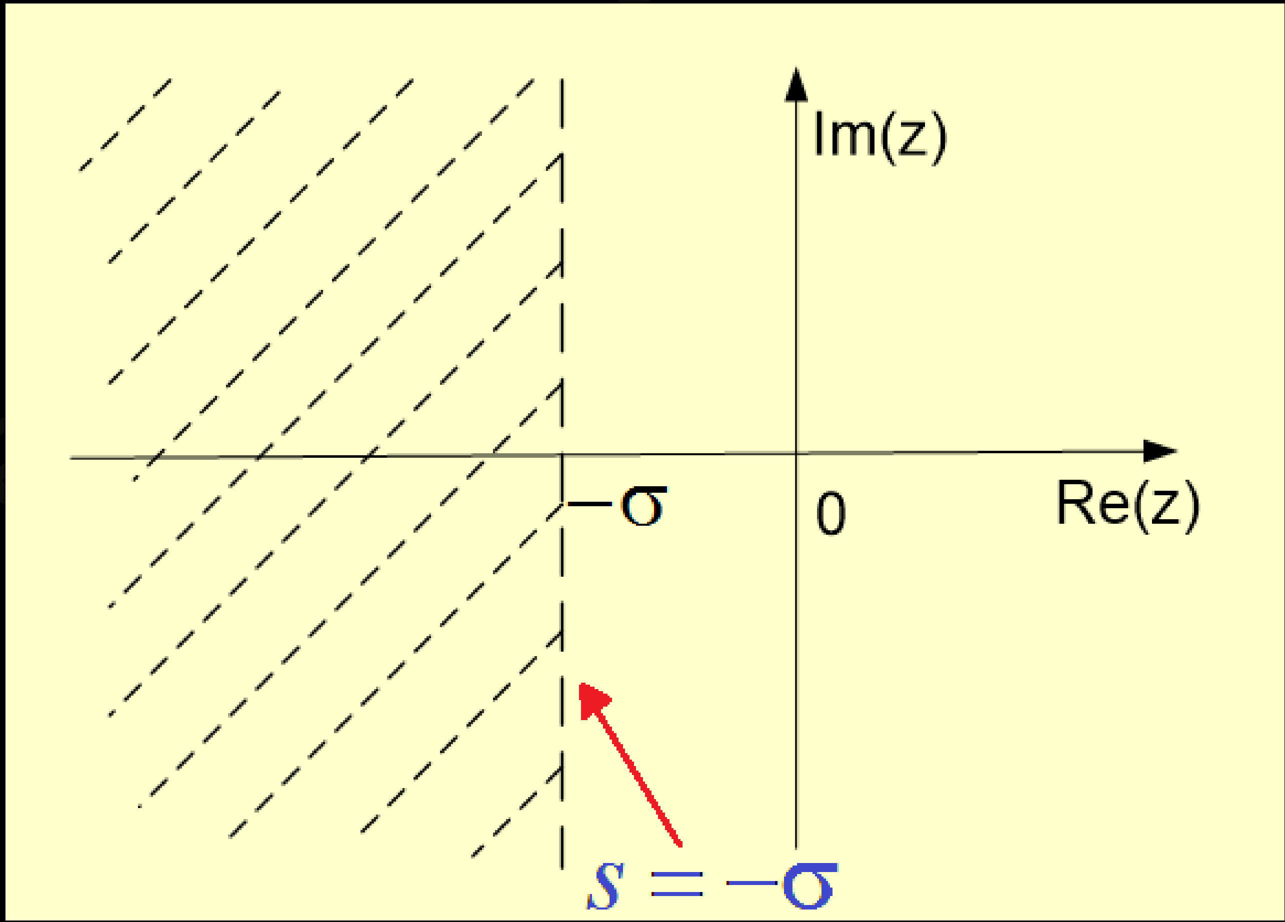
For example:
If in *Routh-Hurwitz table* for $\bar{p}(s)$ there is no *sign change* in the *pivot column* ('*zero change*'), then $p(s)$ will not have *poles* ('*zero poles*') *to the right* of line $s = -\sigma$

Stability

In other words, the *poles* will be located in the region shown in the figure:

to the left of the line $s = -\sigma$, or in the line itself

As $\bar{p}(s)$ is a translation of $p(s)$ by σ units to the *right*, then all the conclusions extracted for $\bar{p}(s)$ with respect to the *imaginary axis* are valid to $p(s)$ with respect to the line $s = -\sigma$



Example 10:

Verify if the *characteristic polynomial* $p(s)$ given below has its poles lying to the *left* of $s = -2$

$$p(s) = s^3 + 8s^2 + 21s + 20$$

Let us verify first if $p(s)$ has all its *roots* in the LHP

$$p(s) = s^3 + 8s^2 + 21s + 20$$

s^3	1	21
s^2	8	20
s^1	18.5	
s^0	20	

zero changes in the *pivot column*

The *pivot column* of *Routh-Hurwitz* table of $p(s)$ is all *positive* and therefore, the *polynomial* has its poles in the LHP

Example 10 (continued):

But, will there be *poles* to the *left* of the line $s = -2$? In order to find this we need to calculate the polynomial $\bar{p}(s)$ with $\sigma = 2$

$$\bar{p}(s) = p(s-2) = (s-2)^3 + 8(s-2)^2 + 21(s-2) + 20$$
$$= s^3 + 2s^2 + s + 2$$

s^3	1	1
s^2	2	2
s^1	$0 \cong \varepsilon > 0$	
s^0	2	

0 ('zero') in the *pivot column*

There is no *sign change* in the *pivot column*

Example 10 (continued):

This means that there is no roots of $\bar{\mathrm{p}}(\mathrm{s})$ in the RHP and then, there is no roots of $\mathrm{p}(\mathrm{s})$ to the *right* of line $\mathrm{s} = -2$

But that '*zero*' in the second line from below (line s^1) in the *pivot column* indicates that there are 2 *roots* of $\bar{\mathrm{p}}(\mathrm{s})$ in the *imaginary axis* and therefore, there are 2 *roots* of $\mathrm{p}(\mathrm{s})$ lying on the line $\mathrm{s} = -2$

Verifying the location of the *roots* of $\mathrm{p}(\mathrm{s})$ we can ratify that:

- there is no *pole* to the *right* of the line $\mathrm{s} = -2$
- there is 1 *pole* to the *left* of the line $\mathrm{s} = -2$ (at $\mathrm{s} = -4$)

and:

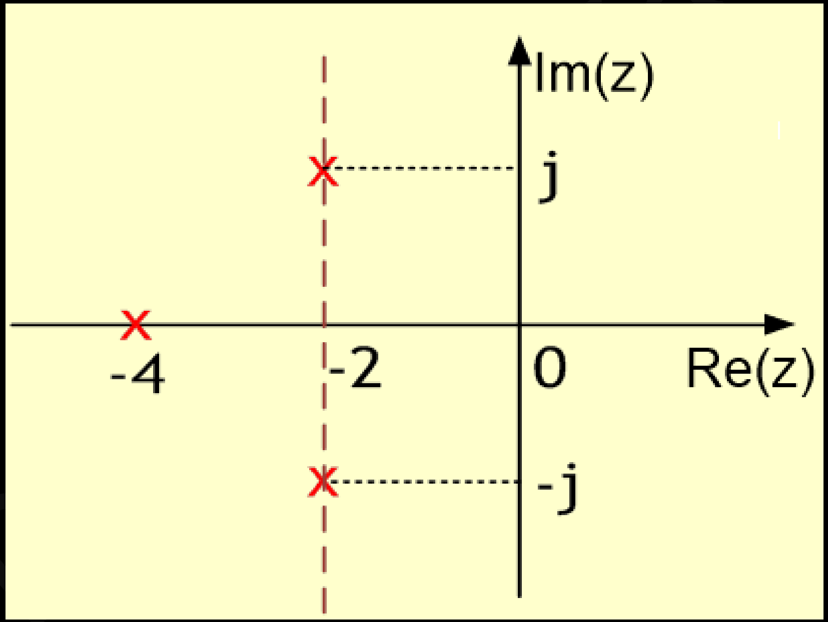
- there are 2 *poles* on the line $\mathrm{s} = -2$

Example 10 (continued):

The *roots* of $p(s)$ are:

$$s = -4$$

$$s = -2 \pm j$$



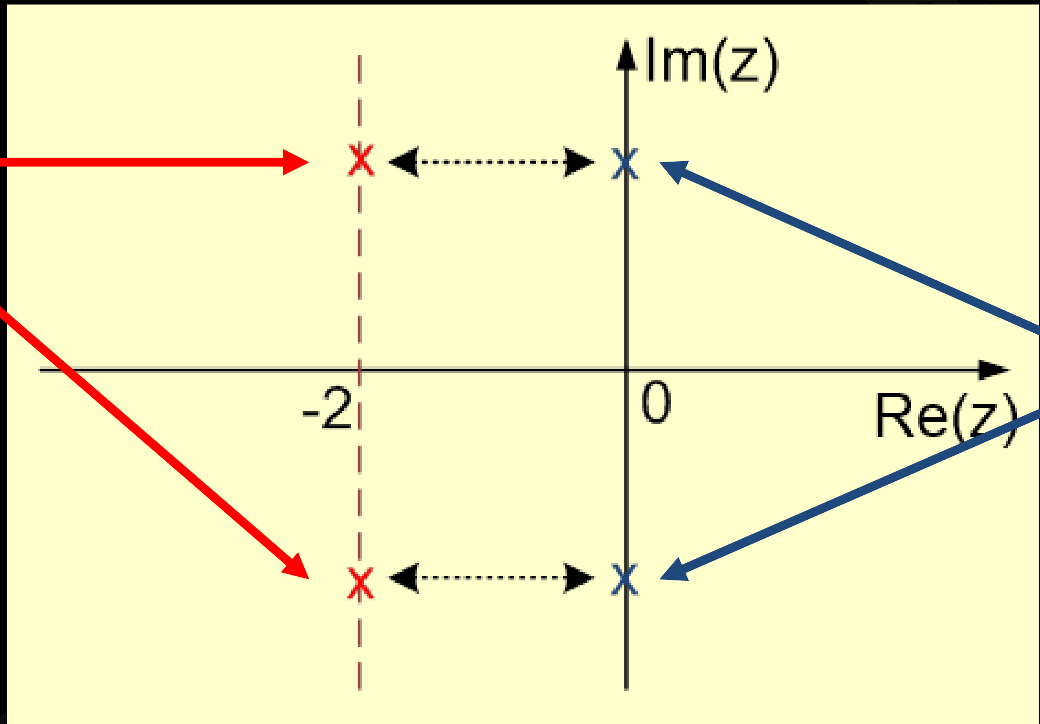
Location of the *roots* of *polynomial* $p(s)$ in the complex plane

Stability

Example 10 (continued):

The 2 *roots* of $\bar{\mathrm{p}}(\mathrm{s})$ in the *imaginary axis* correspond to the 2 *roots* of $\mathrm{p}(\mathrm{s})$ in line $\mathrm{s} = -2$

roots of $\mathrm{p}(\mathrm{s})$ in line $\mathrm{s} = -\sigma$



roots of $\bar{\mathrm{p}}(\mathrm{s})$ in the *imaginary axis*

Location of the 2 *roots* of the polynomials $\bar{\mathrm{p}}(\mathrm{s})$ and $\mathrm{p}(\mathrm{s})$

Example 11:

Verify if the *characteristic polynomial* $p(s)$ given below has its *poles* to the *left* of $s = -1$

$$p(s) = 2s^4 + 13s^3 + 28s^2 + 23s + 6$$

Firstly let us verify if $p(s)$ has all its *roots* located in the LHP

s^4	2	28	6
s^3	13	23	0
s^2	24.46	6	
s^1	19.81	0	
s^0	6		

the *pivot column* is all *positive* and therefore this *polynomial* $p(s)$ has all its *poles* lying in the LHP.

Example 11 (continued):

However, we want to know if they lye to the left of the line $s = -1$

For this we need to calculate the *polynomial* $\bar{p}(s)$ by setting $\sigma = 1$

$$\bar{p}(s) = p(s-1) = 2(s-1)^4 + 13(s-1)^3 + 28(s-1)^2 + 23(s-1) + 6$$
$$= 2s^4 + 5s^3 + s^2 - 2s$$

s^4	2	1	0
s^3	5	-2	
s^2	9/5	0	
s^1	-2		
s^0	0		

0 ('zero') in the *pivot column*

there is 1 *sign change* in the *pivot column*, thus

there is one *root* of $\bar{p}(s)$ in the RHP and therefore,

there is one *root* of $p(s)$ to the *right* of the line $s = -1$

Exemplo 11 (continuação):

But that '*zero*' in the last line (line s^0) of the *pivot column* indicates that there is 1 *root* of $\bar{p}(s)$ lying in the *imaginary axis* (at the *origin*) and therefore, there is 1 *root* of $p(s)$ on $s = -1$

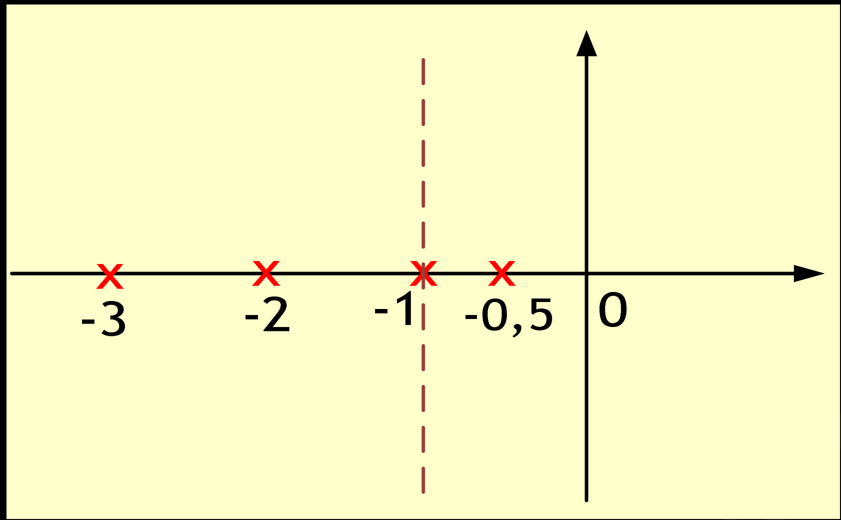
By verifying the *location* of the *roots* of $p(s)$ we can ratify that:

- in fact there is 1 *pole* lying on the line $s = -1$

and that

- There is also 1 *pole* lying to the *right* of the line $s = -1$ (at $s = -0.5$), which obviously is in the interval $\{-1 < s < 0\}$ since we have already seen that $p(s)$ has no *roots* in the RHP

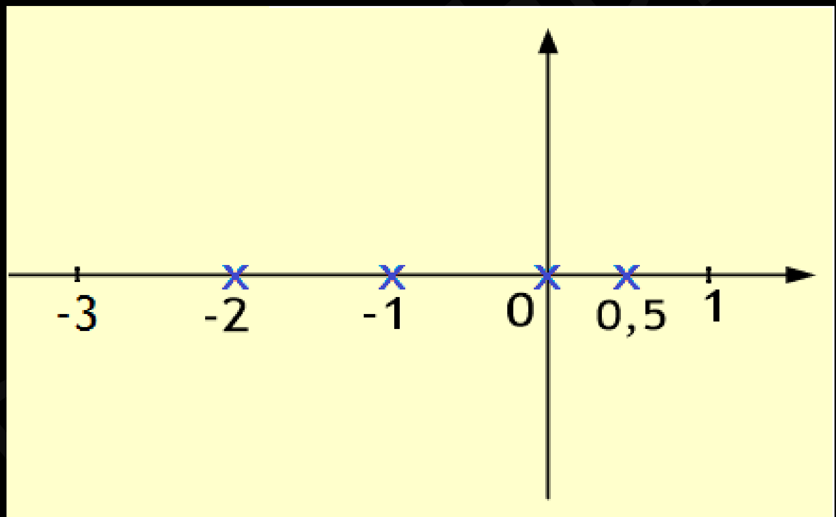
Exemplo 11 (continuação):



The *roots* of $\mathrm{p(s)}$ are:

$$s = -3$$
$$s = -2$$
$$s = -1$$
$$s = -0.5$$

Location of the *roots* of the *polynomial* $\mathrm{p(s)}$



The *roots* of $\bar{\mathrm{p}}(\mathrm{s})$ are:

$$s = -2$$
$$s = -1$$
$$s = 0$$
$$s = 0.5$$

Location of the *roots* of the *polynomial* $\bar{\mathrm{p}}(\mathrm{s})$

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Departamento de
Engenharia Eletromecânica

Thank you!

Obrigado!

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