Control Systems

9

"State Equations"

(part II)

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Recollecting (*from part I*),

the *state equations* have the form (systems of n^{th} order)

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

where:

A is a n x n matrix

B is a n x p matrix

C is a q x n matrix

D is a q x p matrix

with:

p = number of inputs

q = number of outputs

$\dot{x}$ = derivative of vector x

State Equations

$\dot{x}$ = the derivative of vector x

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \qquad \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix}$$

State Equations

For the case of *systems* with only one *input* $u(t)$, i.e., $p = 1$, we have that:

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

that is, in this case

B is a *column vector*.

For the case of *systems* with only one *output* $y(t)$, i.e., $q = 1$, we have that:

$$C = [\, c_1 \quad c_2 \quad \dots \quad c_n \,]$$

C is a *row vector*.

For the case of systems with only one *input* $u(t)$ and one *output* $y(t)$,

$$D = [\, d_1 \,]$$

D is a *constant* d_1 (that is,

D is a $1x1$ matrix).

A *matrix* A in the "*companion form*" has the following aspect:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \left(\frac{-a_n}{a_o}\right) & \left(\frac{-a_{n-1}}{a_o}\right) & \left(\frac{-a_{n-2}}{a_o}\right) & \left(\frac{-a_{n-3}}{a_o}\right) & \cdots & \left(\frac{-a_1}{a_o}\right) \end{bmatrix}$$

where, a_o , a_1 , … , a_{n-1} , a_n , are the *coefficients* of the *characteristic equation* $p(s)$:

$$p(s) = a_o s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

In the particular case, but very common, of $a_o = 1$, matrix A in the "*companion form*" has the following aspect:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}$$

where, a_1 , … , a_{n-1} , a_n , are the *coefficients* of the *characteristic equation* $p(s)$:

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

The *characteristic equation* and the *poles* of the system

A system described in the form of *state equations*

$$\begin{cases} \dot{x} = A x + B u \\ y = C x + D u \end{cases}$$

has its characteristic *polynomial* given by:

$$p(s) = \det\{[sI - A]\}$$

The *poles* of the system are the "*eigenvalues*" of A, which can be *repeated*, i.e., *double*, *triple*, etc.

It is well known that the *eigenvalues* of A are the *roots* of the *characteristic polynomial*

$$p(s) = \det[\, s \cdot I - A \,]$$

Equivalent representations

Consider a system described by *state equations*

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

which the *state variable* is $x(t)$

Now defining a new *state variable* $\bar{x}$ as:

$$\bar{x} = P\,x$$

P being *invertible*.

thus, since:

$$\dot{\bar{x}} = P\,\dot{x}$$

we have that:

$$\begin{cases} x = P^{-1}\,\bar{x} \\ \dot{x} = P^{-1}\,\dot{\bar{x}} \end{cases}$$

that is:

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,u \\ y = \bar{C}\,\bar{x} + \bar{D}\,u \end{cases}$$

this is another representation of the same system in *state equations*

where:

$$\begin{cases} \bar{A} = P\,A\,P^{-1} \\ \bar{B} = P\,B \\ \bar{C} = C\,P^{-1} \\ \bar{D} = D \end{cases}$$

Note that the *input* u and the *output* y do not change.

Only the *internal representation* of the system (as *state variable*).

Conversion from the State Equation to Transfer Function

In order to convert the representation of a system in *state equations*

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

to *transfer function*, the expression is given by,

$$\frac{Y(s)}{U(s)} = C \cdot (sI - A)^{-1} \cdot B + D$$

analog simulation

Analog Simulation

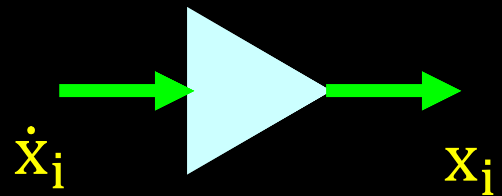
No matter what is the nature of a *linear time invariant* (LTI) system (mechanical, electrical, electromechanical, thermic, hydraulic, or a chemical process, etc.) it can be simulated in laboratory by using electronic components.

In that way it is possible to simulate the *input* of any *system*, such as for example a step function, and to observe what it would be the *output* of the *system* for that *input*.

That is called "*analog simulation*".

Components with which we do the *analog simulation*

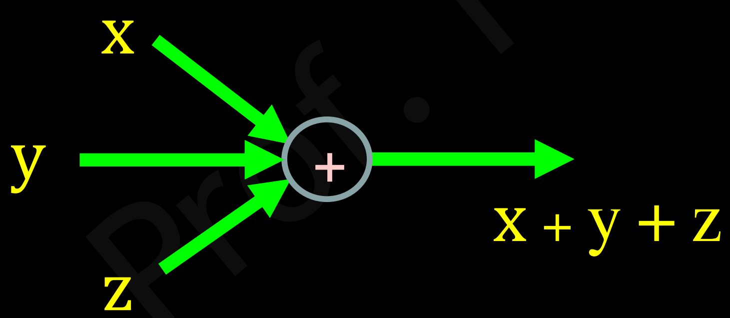
Integrator



transforms an input signal $\dot{x}_i$ in x_i in its output, that is, it integrates

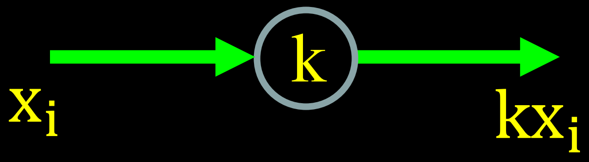
The *analog simulation* of a n order *system* needs n integrators.

Adding element



sums the input signals giving an output signal

MULTIPLIER

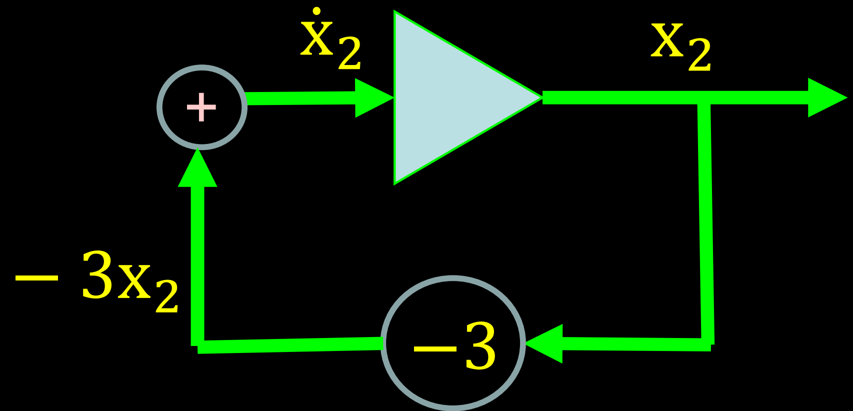


multiplies the *input* signal x_i by k, giving back kx_i in its *output*

Example 16:

In the figure below we can see how it is done the *analog simulation* of the *differential equation*

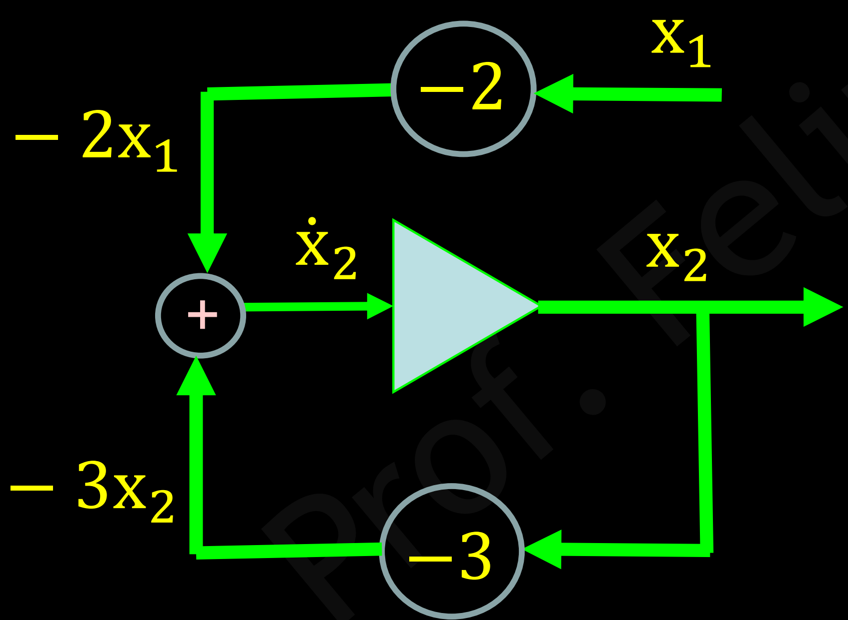
$$\dot{x}_2 = -3x_2$$



Example 17:

Now the *analog simulation* of the *differential equation*

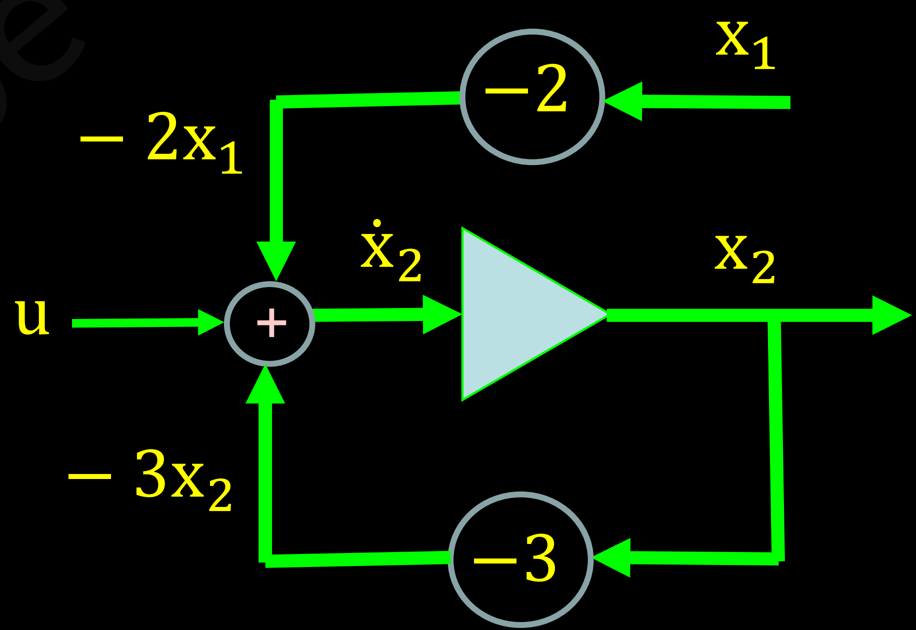
$$\dot{x}_2 = -2x_1 - 3x_2$$



Example 18:

And now the *analog simulation* of the *differential equation*

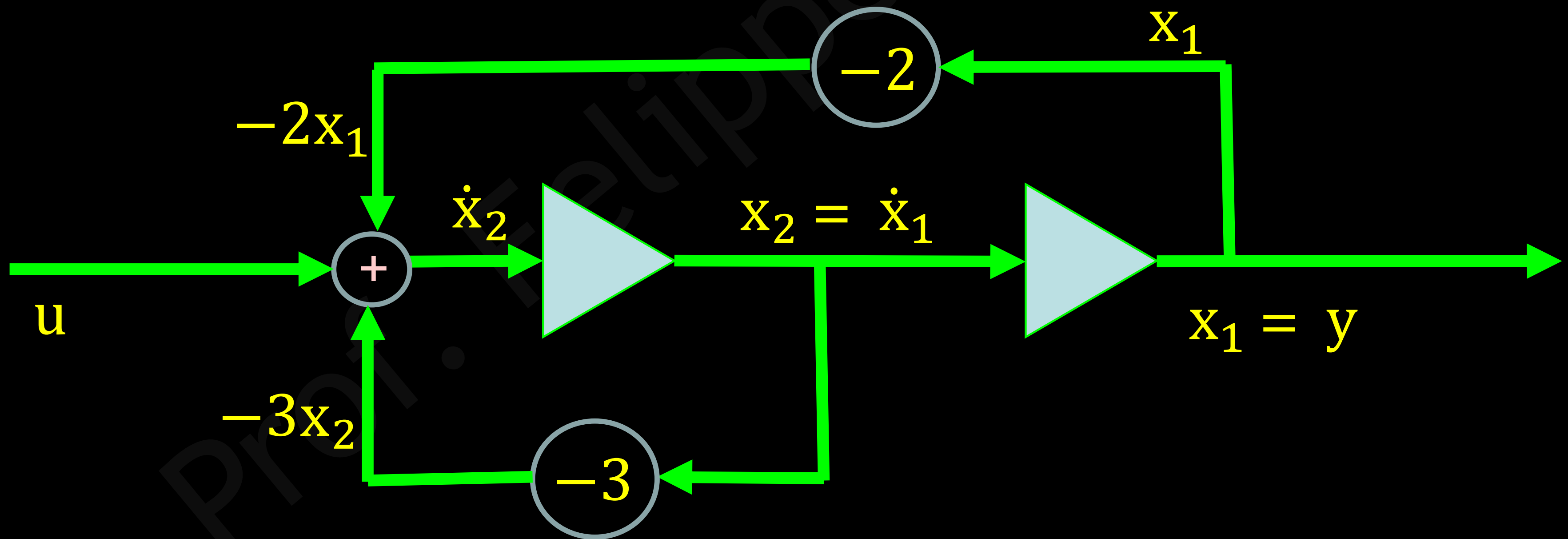
$$\dot{x}_2 = -2x_1 - 3x_2 + u$$



Example 19:

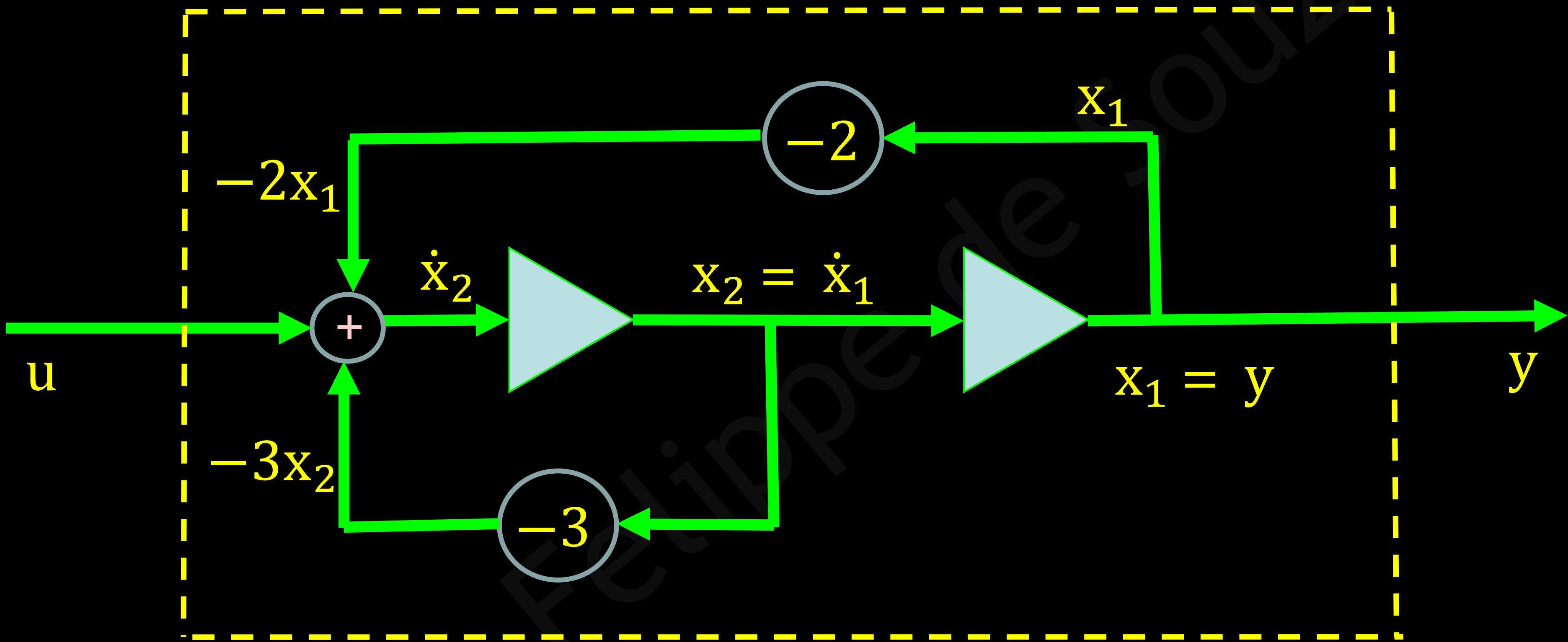
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 3x_2 + u \\ y = x_1 \end{cases}$$

Let us now do the *analog simulation* of this 2nd order *system* (described by its *state equations*)



Example 19 (continued):

Now, if we draw a box covering the *analog simulation* done



we can observe that in this box only the *input* u comes in and only the *output* y comes out.

The variables x_1 and x_2 which stayed inside the box are part of an internal representation of the *system*, though its *state variable*

Analog Simulation

in practice

conversion of the transfer function
to state equations

Conversion of the Transfer Function to State Equations

We have seen the representation of a system by its *transfer function*

$$\frac{Y(s)}{U(s)}$$ is unique!

On the other hand the representation of a system in *state equations*

$$\begin{cases} \dot{x} = A x + B u \\ y = C x + D u \end{cases}$$

It is not unique!

There is no single rule to transform systems described by its *ordinary differential equation* (ODE) or by its *transfer function*, in *state equations*

Let us see the same *third order* system A, described by the *differential equation*

$$\dddot{y} + 12\ddot{y} + 20\dot{y} = 80u$$

which its *transfer function* is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{80}{s^3 + 12s^2 + 20s}$$

or

$$G(s) = \frac{Y(s)}{U(s)} = \frac{80}{s(s+2)(s+10)}$$

Example 20:

For system A described by the *ordinary differential equation* (ODE) is given by (1)

$$\dddot{y} + 12\ddot{y} + 20\dot{y} = 80u$$

Defining the *state variable*

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \end{cases}$$

we get the *state equations* ⇒

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -20x_2 - 12x_3 + 80u \\ y = x_1 \end{cases}$$

Example 20 (continued):

and writing in the matrix form we get

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -12 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 80 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x \end{cases}$$

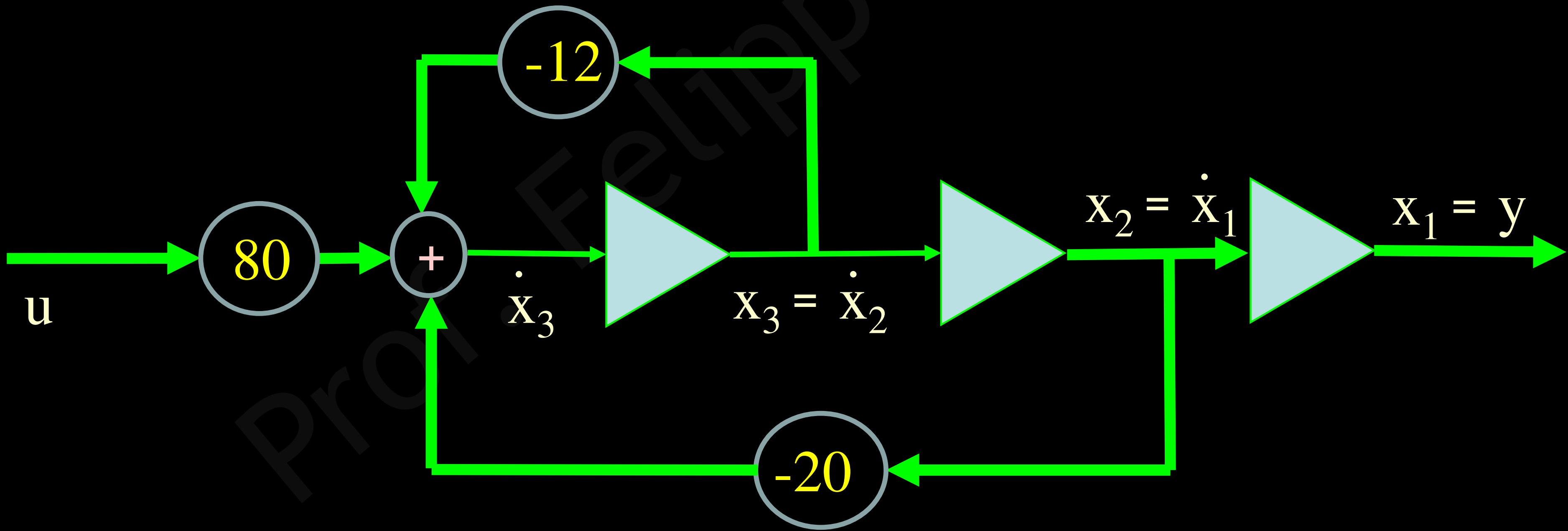
$D = 0$

Note that matrix A is in the *companion form*

Example 21:

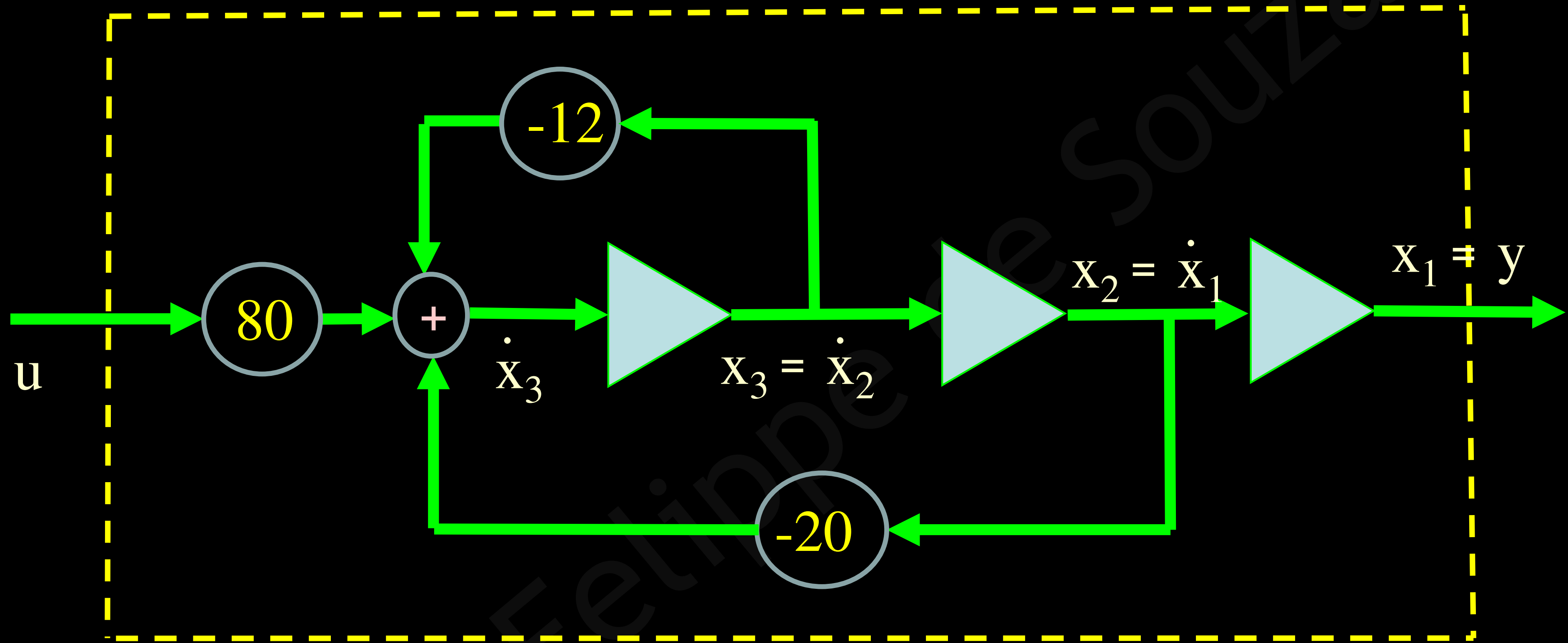
Let us now do a *analog simulation* of this system A using the *state equation* obtained in the previous example

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -20x_2 - 12x_3 + 80u \\ y = x_1 \end{cases}$$



Example 20 (continued):

Now, by drawing a box covering the *analog simulation* done



we can observe that in this box only the *input* u comes in and only the *output* y comes out.

The variables x_1 and x_2 which stayed inside the box are part of an *internal representation* of the *system*, though its *state variable*

Example 22:

Let us consider again the same system A of the previous example. However, here we are going to rewrite the *transfer function* $\mathrm{G(s)}$ given by (2) in the following form:

$$\mathrm{G(s)} = \underbrace{\frac{5}{s}}_{\frac{X_1(s)}{U(s)}} \cdot \frac{4}{(s+2)} \cdot \frac{4}{(s+10)} = \frac{Y(s)}{U(s)}$$

Defining the *state variable* in the following form:

$$\frac{X_1(s)}{U(s)} = \frac{5}{s}, \quad \frac{X_2(s)}{U(s)} = \frac{5}{s}\cdot\frac{4}{(s+2)}, \quad \frac{X_3(s)}{U(s)} = \frac{5}{s}\cdot\frac{4}{(s+2)}\cdot\frac{4}{(s+10)}$$

Example 22 (continued):

$$\begin{cases} X_1(s) = \dfrac{5\ U(s)}{s} \\[2ex] X_2(s) = \dfrac{20\ U(s)}{s(s+2)} \\[2ex] X_3(s) = G(s).U(S) = Y(s) = \dfrac{80\ U(s)}{s(s+2)(s+10)} \end{cases}$$

$$\begin{cases} sX_1(s) = 5U(s) \\[2ex] (s+2)\ X_2(s) = 4.\dfrac{5\ U(s)}{s} = 4.X_1(s) \\[2ex] (s+10)\ X_3(s) = 4.\dfrac{20\ U(s)}{s(s+2)} = 4.X_2(s) \end{cases}$$

Example 22 (continued):

$$\begin{cases} \dot{x}_1 = 5u \\ \dot{x}_2 = 4x_1 - 2x_2 \\ \dot{x}_3 = 4x_2 - 10x_3 \\ y = x_3 \end{cases}$$

Which give us a second *formulation* in *state equations* for this system A, different from the *formulation* of the previous example

Example 22 (continued):

by writing in the matrix form we have

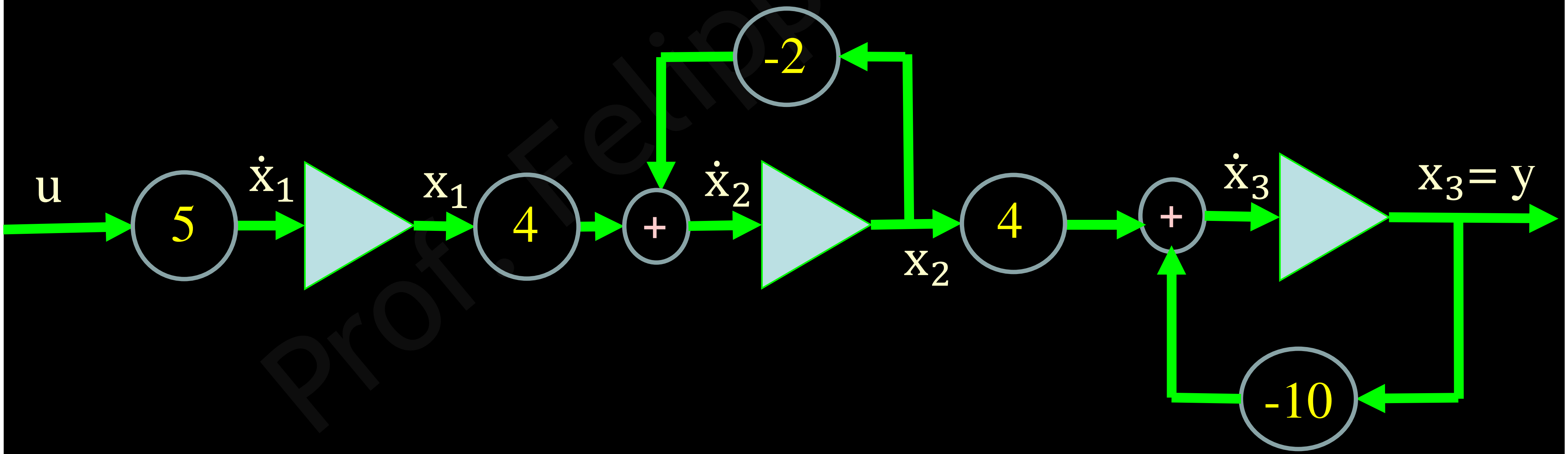
$$\begin{cases} \dot{x} = \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 4 & -2 & 0 \\ 0 & 4 & -10 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}}^{B} \\ y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

Example 23:

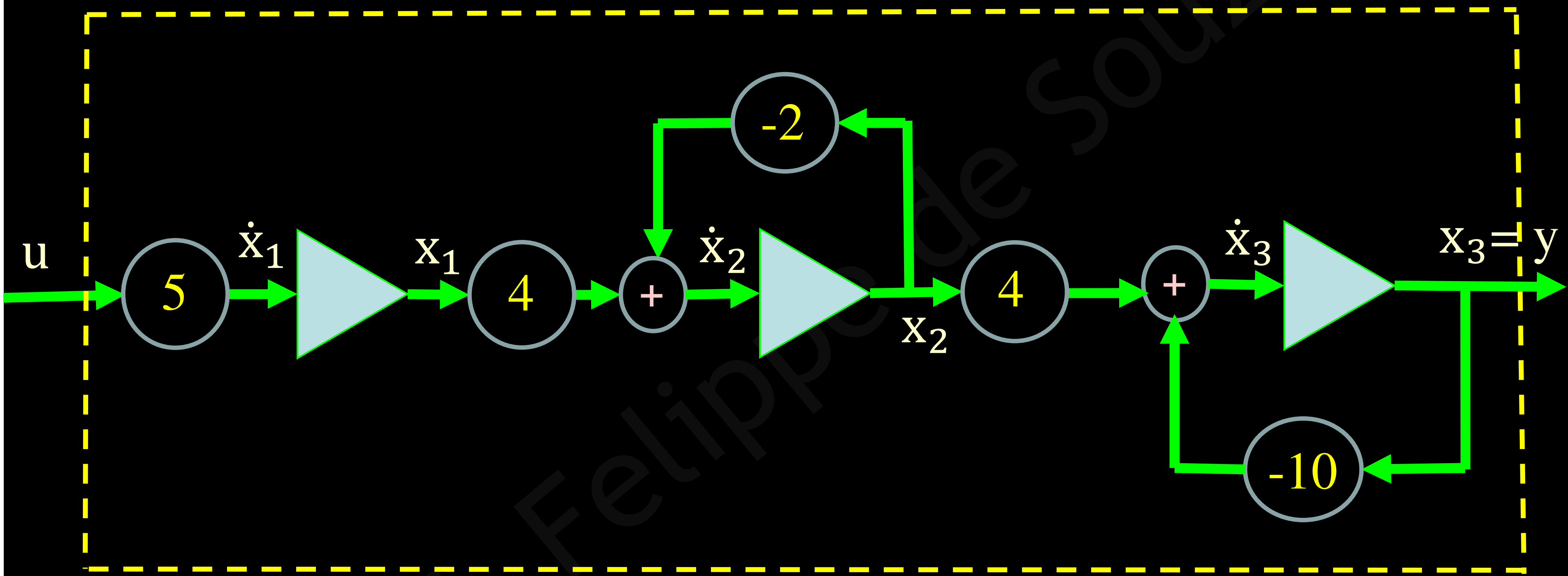
Let us now do a *analog simulation* of this system $\underline{A}$ using the *state equation* obtained in the previous example

$$\begin{cases} \dot{x}_1 = 5u \\ \dot{x}_2 = 4x_1 - 2x_2 \\ \dot{x}_3 = 4x_2 - 10x_3 \\ y = x_3 \end{cases}$$



Example 23 (continued):

Again, by drawing a box covering the *analog simulation* done



then we can observe that in this box only the *input* u comes in and only the *output* y comes out.

The variables x_1 and x_2 which stayed inside the box are part of an *internal representation* of the *system*, though its *state variable*

Example 24:

Let us consider again the same system $\underline{A}$ of the 2 previous examples

However, here we are going to rewrite the *transfer function* $G(s)$ given in (3) by expanding in *partial fractions* and defining the *state variables* $X_1(s)$, $X_2(s)$ and $X_3(s)$ of the form shown below:

$$G(s) = \underbrace{\frac{4}{s}}_{\frac{X_1(s)}{U(s)}} + \underbrace{\frac{-5}{(s+2)}}_{\frac{X_2(s)}{U(s)}} + \underbrace{\frac{1}{(s+10)}}_{\frac{X_3(s)}{U(s)}} = \frac{Y(s)}{U(s)}$$

Example 24 (continued):

we then have

$$\begin{cases} X_1(s) = \dfrac{4U(s)}{s} \\[2ex] X_2(s) = \dfrac{-5U(s)}{(s+2)} \\[2ex] X_3(s) = \dfrac{U(s)}{(s+10)} \\[2ex] Y(s) = \left[\dfrac{X_1(s)}{U(s)} + \dfrac{X_2(s)}{U(s)} + \dfrac{X_3(s)}{U(s)}\right] U(s) \end{cases} \Rightarrow \begin{cases} sX_1(s) = 4U(s) \\ sX_2(s) = -2X_2(s) - 5U(s) \\ sX_3(s) = -10X_3(s) + U(s) \\ Y(s) = X_1(s) + X_2(s) + X_3(s) \end{cases}$$

Example 24 (continued):

$$\begin{cases} \dot{x}_1 = 4u \\ \dot{x}_2 = -2x_2 - 5u \\ \dot{x}_3 = -10x_3 + u \\ y = x_1 + x_2 + x_3 \end{cases}$$

So, we have got a third representation in *state equations* for this same system A, different the previous ones.

Writing in matrix form

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{C} x \end{cases}$$

D = 0

State Equations

Example 24 (continued):

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

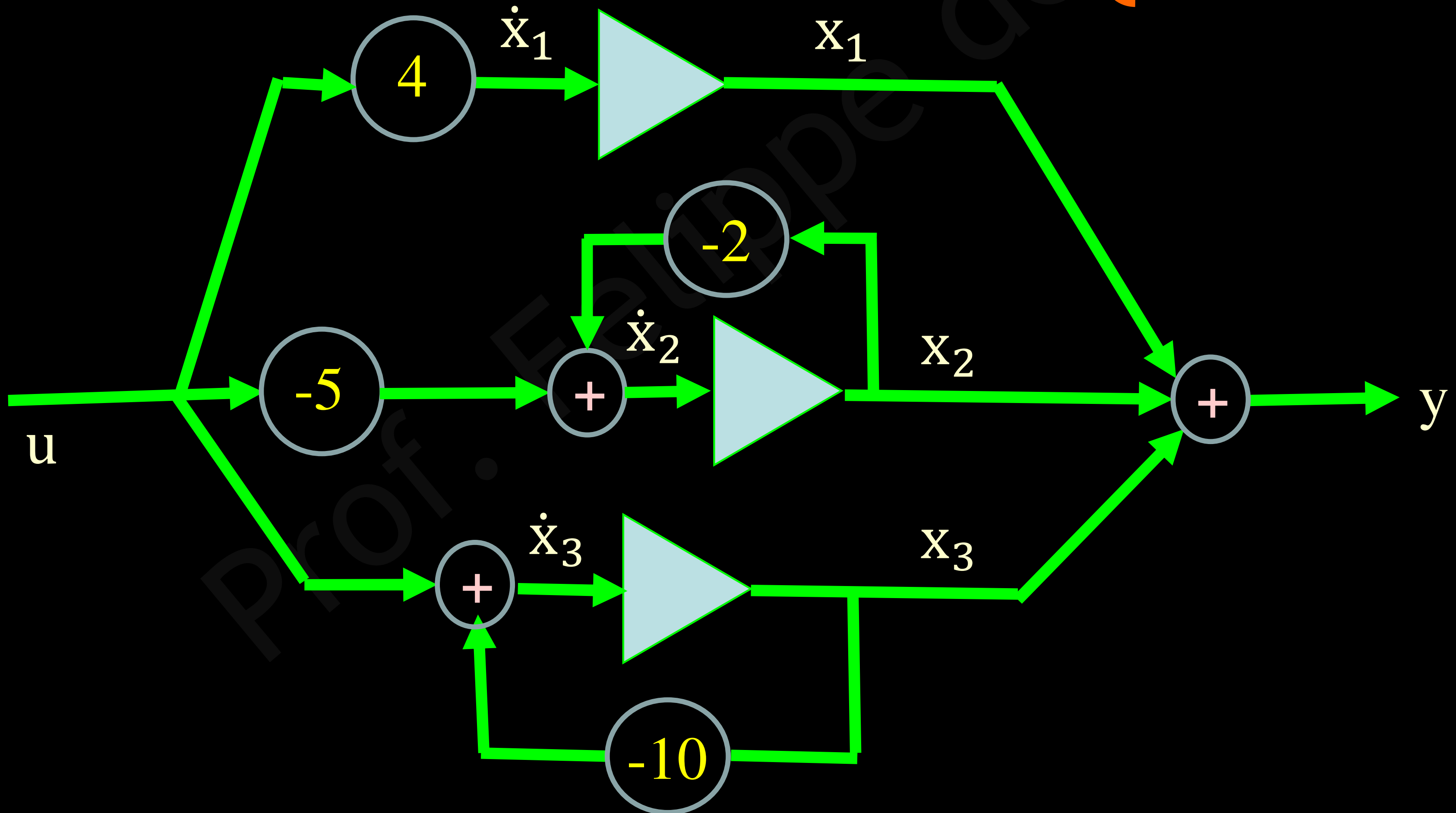
Note that *matrix* A is in the diagonal form in this *representation* and the *poles* of the system ($s = 0$, $s = -2$ and $s = -10$) are the elements of the *main diagonal*

It is obvious that this happens: since *matrix* A is diagonal, then the *elements* of its *main diagonal* are the own *eigenvalues* of the system.

Example 25:

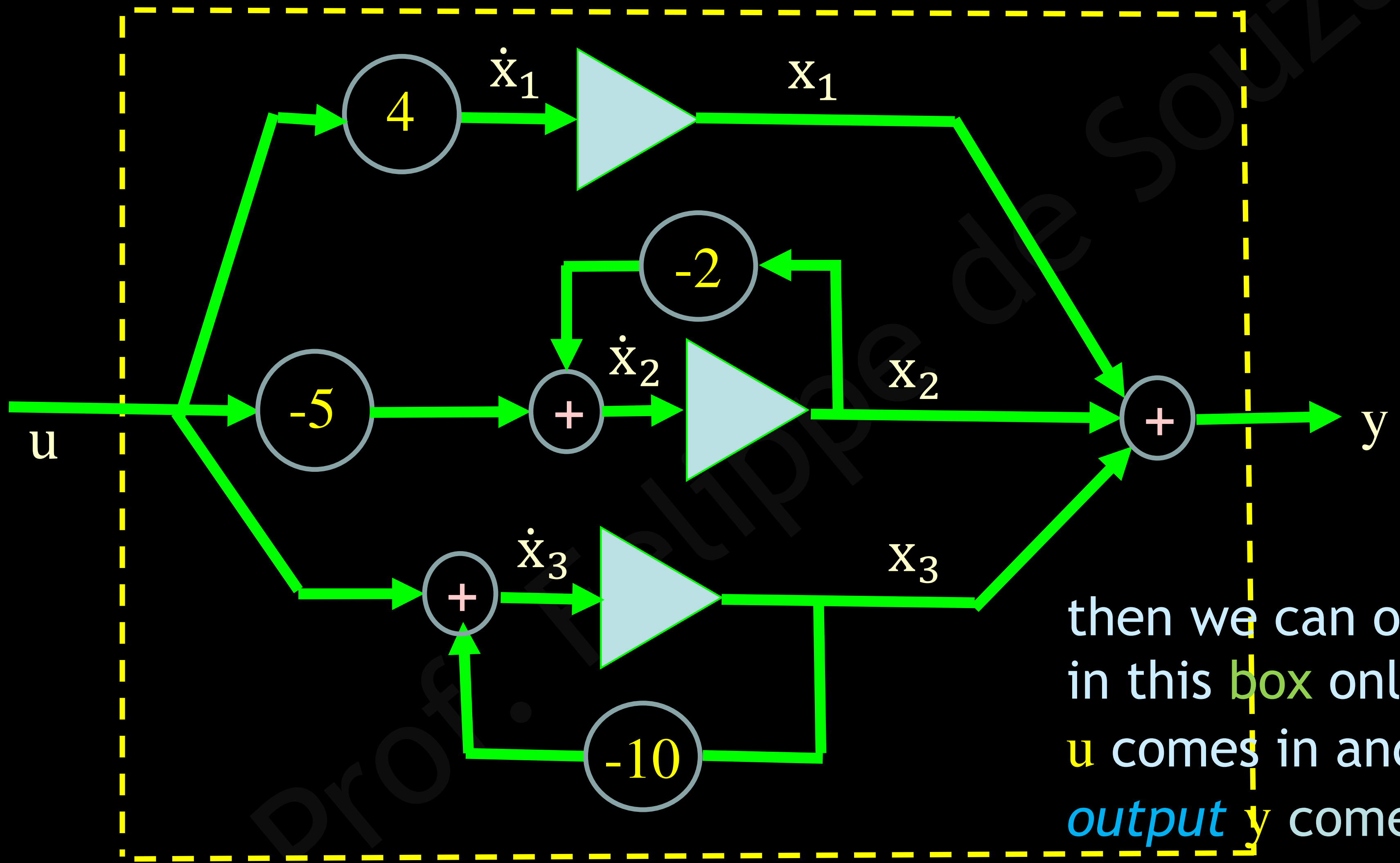
Let us now do a *analog simulation* of this system A using the *state equation* obtained in the previous example

$$\begin{cases} \dot{x}_1 = 4u \\ \dot{x}_2 = -2x_2 - 5u \\ \dot{x}_3 = -10x_3 + u \\ y = x_1 + x_2 + x_3 \end{cases}$$



Example 25 (continued):

Once again, by drawing a box covering the *analog simulation* done



then we can observe that in this box only the *input* u comes in and only the *output* y comes out.

The variables x_1 and x_2 which stayed inside the box are part of an *internal representation* of the *system*, though its *state variable*.

State Equations

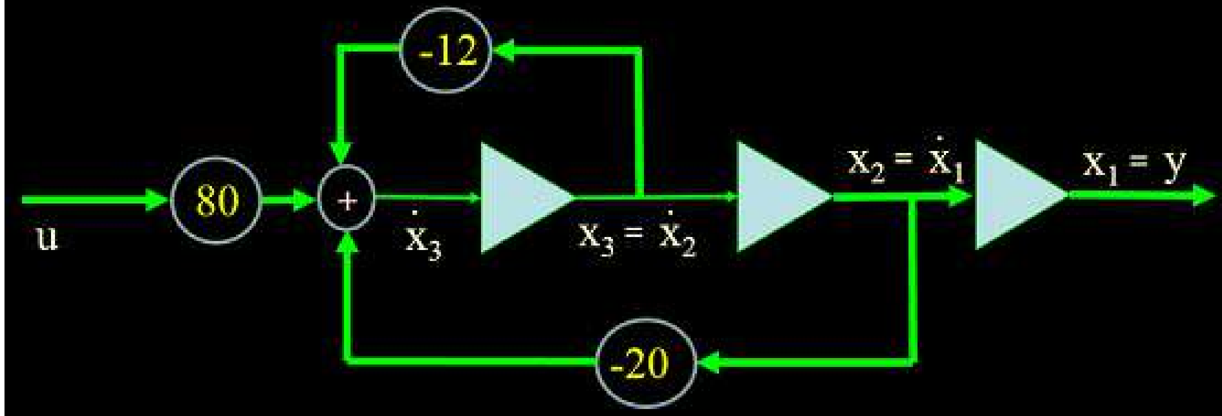
In the previous examples we have obtained 3 different *representations* in *state equations* for the same *system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -20x_2 - 12x_3 + 80u \\ y = x_1 \end{cases}$$

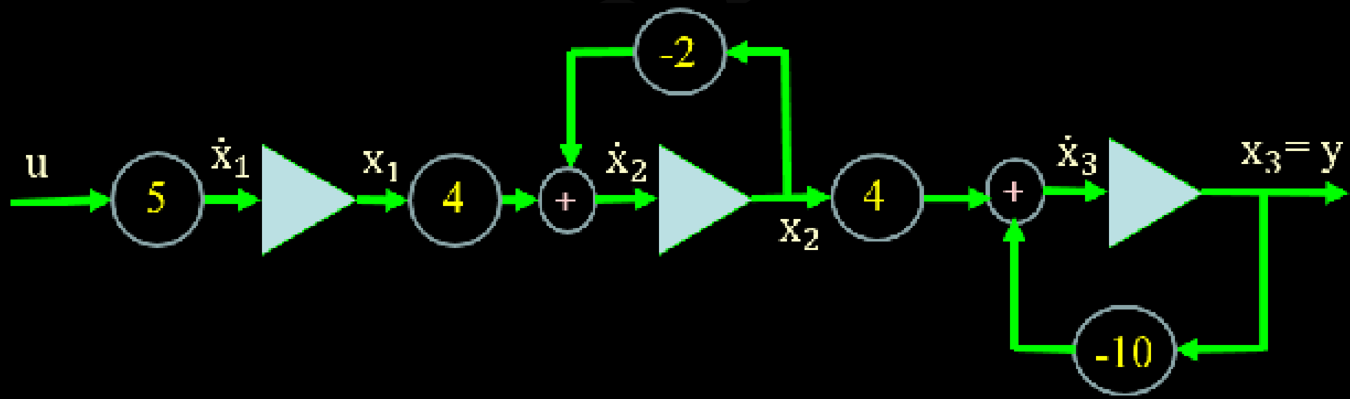
$$\begin{cases} \dot{x}_1 = 5u \\ \dot{x}_2 = 4x_1 - 2x_2 \\ \dot{x}_3 = 4x_2 - 10x_3 \\ y = x_3 \end{cases}$$

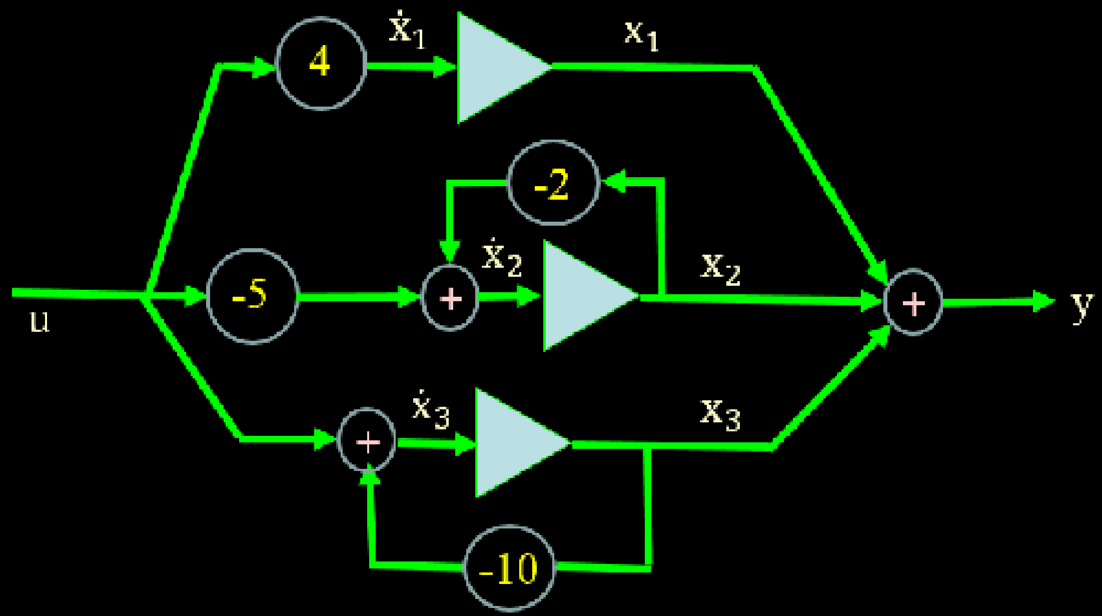
$$\begin{cases} \dot{x}_1 = 4u \\ \dot{x}_2 = -2x_2 - 5u \\ \dot{x}_3 = -10x_3 + u \\ y = x_1 + x_2 + x_3 \end{cases}$$

State Equations



as well as we have obtained 3 different *analog simulations* for the same *system*





As we have already seen in the sections

"Equivalent representations",

the representation of a system in *state equations*

It is not unique!

If the *state variable* is $x(t)$, then for every *invertible* matrix P, we can get a new *state variable*

$$\bar{x}(t) = P\,x(t)$$

and thus, a new *representation* of the *system* in *state equations*

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,u \\ y = \bar{C}\,\bar{x} + \bar{D}\,u \end{cases}$$

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Thank you!

Obrigado!

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