Control Systems

8

"State Equations"

(part I)

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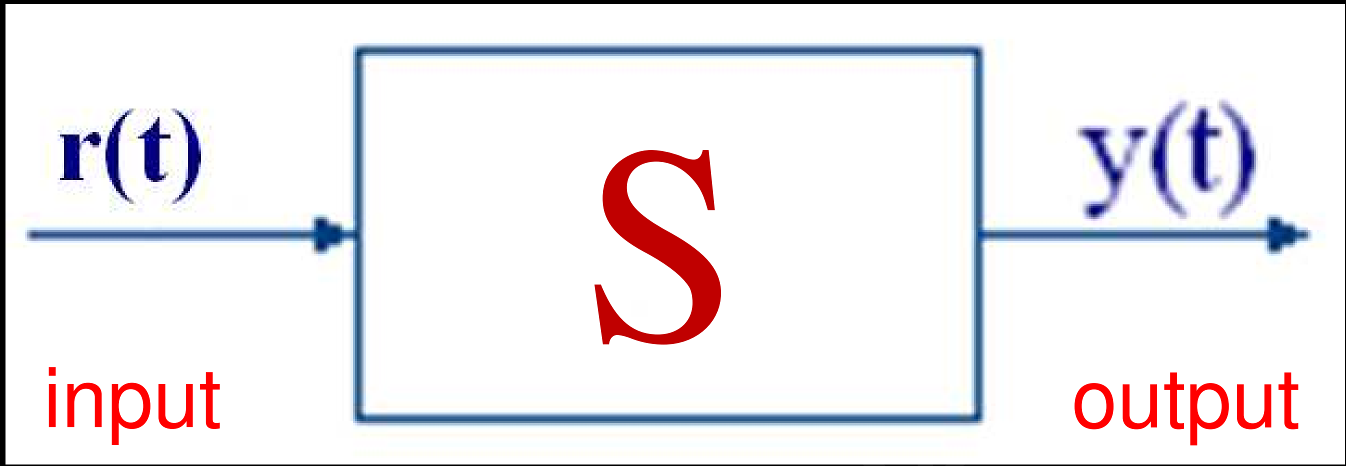
State Equations



We have seen in chapter 4 ("*Systems Representation*") a form of representing *linear* and *time invariant* (*LTI*) systems using *transfer functions* that relates directly the *input* with the output.

Here we will see another form of representing systems by using *internal variables* (*state variable*).
With the *state variables* we can build a system of 1st order *differential equations* that are called "*state equations*".

State Equations

The *representation* of a system in *state equations* considers *internal variables* (*state variables*)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

← "state variable"

called the "*state*".

Normally a "*state* vector" x has n components, where n is the order of the system

The dimension of the *state* vector x can *eventually* be greater than the *order of the system*, but in this case there will be redundant equations.

State Equations

For *linear time invariant* (*LTI*) systems of n^{th} order, the *state equations* have the form:

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

where:

A is a $n \times n$ matrix

B is a $n \times p$ matrix

C is a $q \times n$ matrix

D is a $q \times p$ matrix

with:

p = number of inputs

q = number of outputs

$\dot{x}$ = derivative of vector x

State Equations

$\dot{x}$ = the derivative of vector x

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \qquad \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix}$$

State Equations

For the case of systems with only one *input* $u(t)$, i.e., $p = 1$, we have that:

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

that is, in this case

B is a *column vector*.

For the case of systems with only one *output* $y(t)$, i.e., $q = 1$, we have that:

$$C = [\, c_1 \quad c_2 \quad \dots \quad c_n \,]$$

C is a *row vector*.

For the case of *systems* with only one *input* $u(t)$ and one *output* $y(t)$,

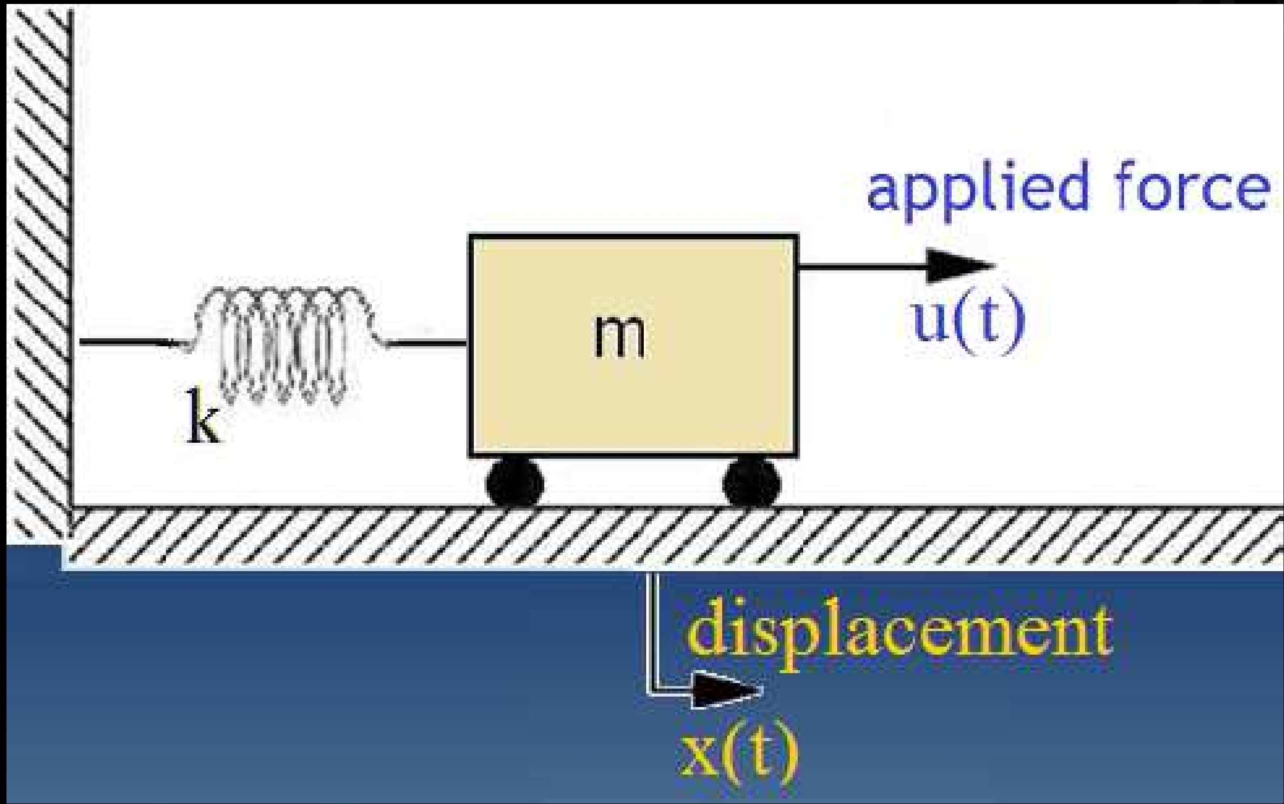
$$D = [\, d_1 \,]$$

D is a *constant* d_1 (that is,

D is a $1x1$ matrix).

Example 1:

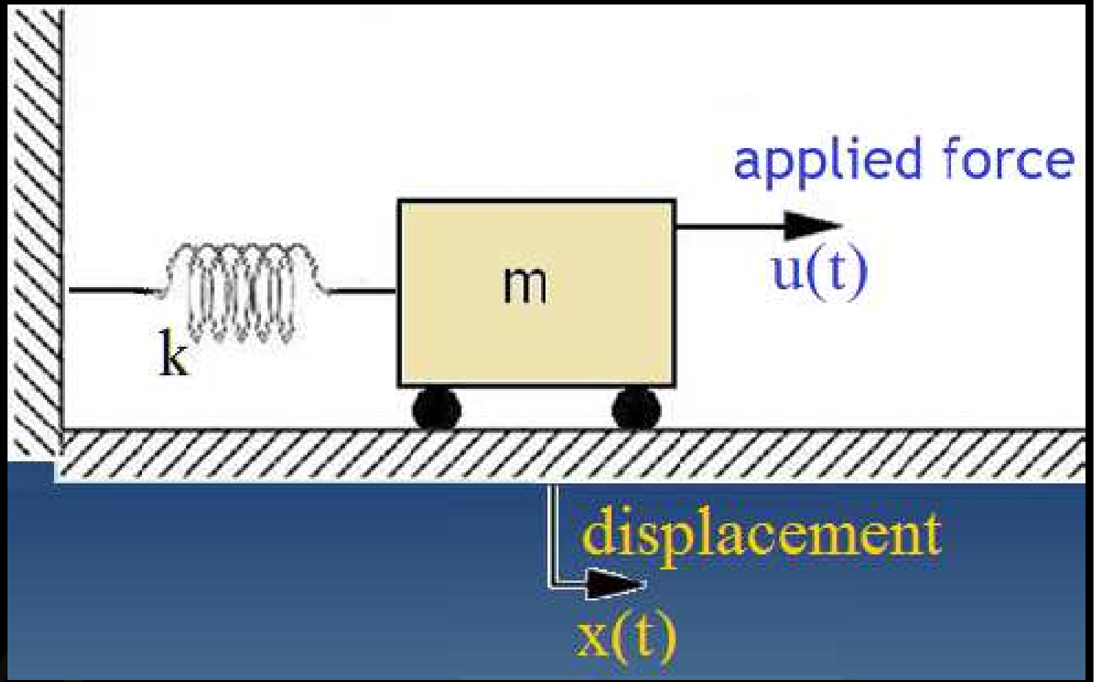
System *cart-mass-spring*



The *ordinary differential equation* (ODE) that describes this system, as seen in chapter 3 (“*System Modelling*”) is given by:

$$my'' + \mu y' + ky = u$$

Example 1 (continued):



Defining the state variable

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where:

$x_1(t) = y(t) =$ *position of the cart* at instant t

$x_2(t) = y'(t) =$ *velocity of the cart* at instant t

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

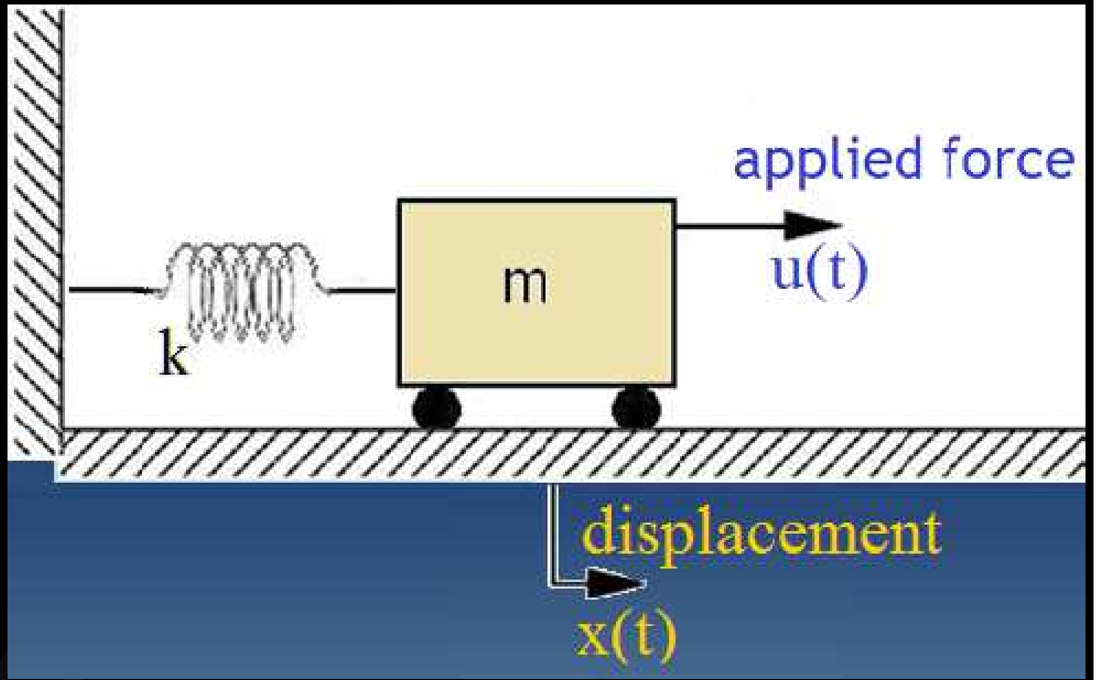
represents the *internal state of the system*.

State Equations

Example 1 (continued):

for example,

$$x(0) = x_o = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$



This means that in the instant of time $t = 0$

the "*state*" of the system: the cart is passing by the origin

(that is, $x_1(0) = 0$)

with velocity – 3m/s,

(that is, 3m/s , $x_2(0) = -3$).

Example 1 (continued):

Then

$$\begin{cases} \dot{x}_1 = y' = x_2 \\ \dot{x}_2 = y'' = -\frac{k}{m}y - \frac{\mu}{m}y' + \frac{1}{m}u \end{cases}$$

and since $x_1 = y$ and $x_2 = y'$ then:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{\mu}{m}x_2 + \frac{1}{m}u \\ y = x_1 \end{cases}$$

Example 1 (continued):

therefore:

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -\mu/m \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

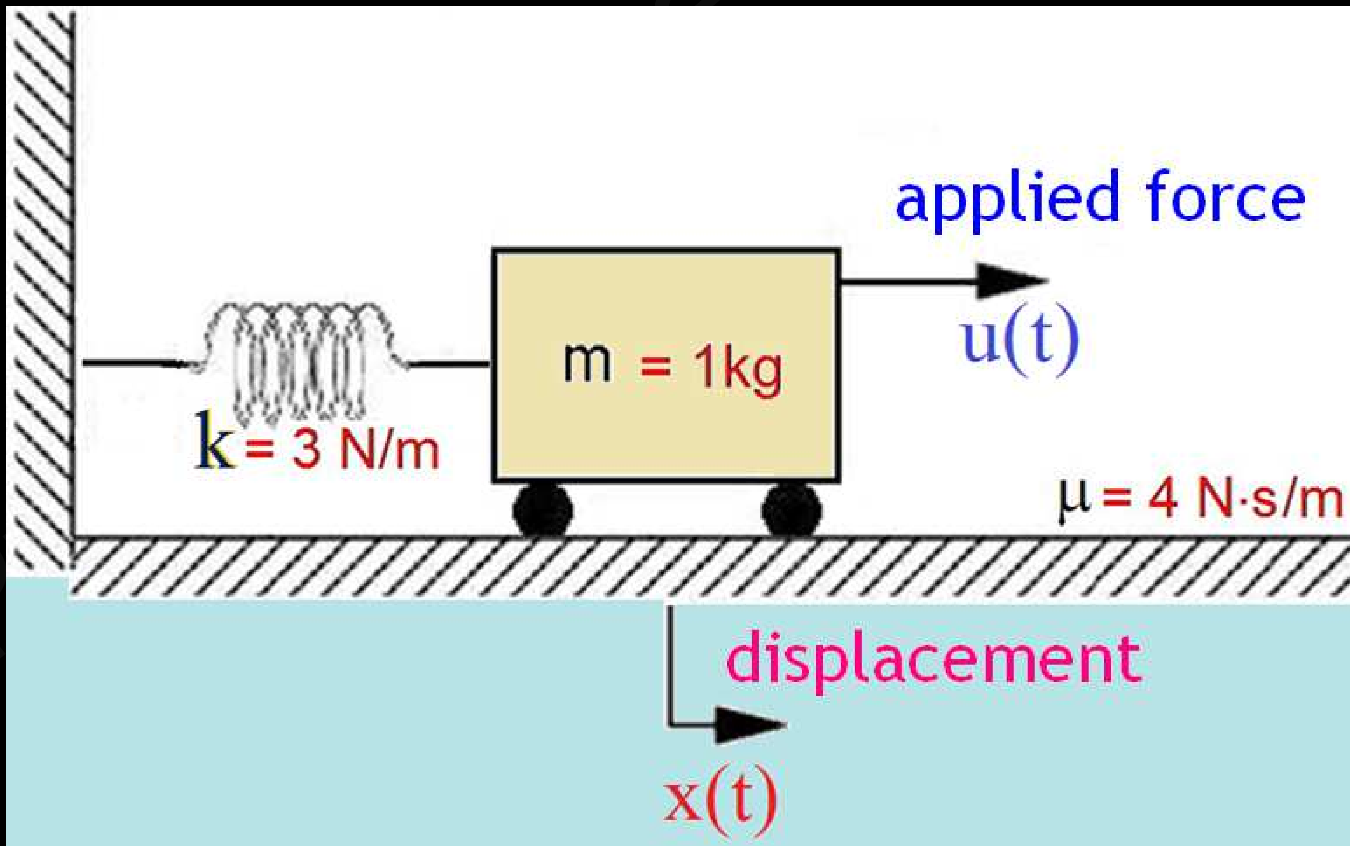
which is the *representation* of the system in *state equations*.

Note that in this case $D = 0$.

Example 2:

System *cart-mass-spring* of previous example with

$$\begin{cases} m = 1 \\ \mu = 4 \\ k = 3 \end{cases}$$



Example 2 (continued):

thus:

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

and hence:

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = [\,0\,]$$

Example 3:

Consider the system described by:

$$y''' + 4y'' + 5y' = 3u$$

which the *transfer function* is given by:

$$\frac{Y(s)}{U(s)} = \frac{3}{s\cdot(s^2 + 4s + 5)}$$

$$= \frac{3}{s^3 + 4s^2 + 5s}$$

Example 3 (continued):

Defining the *state variable* as:

$$\begin{cases} X_1(s) = Y(s) = \dfrac{3 \cdot U(s)}{s^3 + 4s^2 + 5s} \\ X_2(s) = s \cdot Y(s) \\ X_3(s) = s^2 \cdot Y(s) \end{cases}$$

$$\begin{cases} s \cdot X_1(s) = X_2(s) \\ s^2 \cdot X_1(s) = X_3(s) \\ s^3 \cdot X_1(s) + 4\,s^2 \cdot X_1(s) + 5\,s \cdot X_1(s) = 3 \cdot U(s) \end{cases}$$

$sX_2(s)$

$sX_3(s)$

Example 3 (continued):

thus:

$$
\begin{cases}
s \cdot X_1(s) = X_2(s) \\
s \cdot X_2(s) = X_3(s) \\
s \cdot X_3(s) + 4\,\underbrace{s^2 \cdot X_1(s)}_{X_3(s)} + 5\,\underbrace{s \cdot X_1(s)}_{X_2(s)} = 3 \cdot U(s)
\end{cases}
$$

$$
\begin{cases}
s \cdot X_1(s) = X_2(s) \\
s \cdot X_2(s) = X_3(s) \\
s \cdot X_3(s) + 5 \cdot X_2(s) + 4 \cdot X_3(s) = 3 \cdot U(s) \\
Y(s) = X_1(s)
\end{cases}
$$

Example 3 (continued):

$$\begin{cases} s \cdot X_1(s) = X_2(s) \\ s \cdot X_2(s) = X_3(s) \\ s \cdot X_3(s) = -5 \cdot X_2(s) - 4 \cdot X_3(s) + 3 \cdot U(s) \\ Y(s) = X_1(s) \end{cases}$$

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

Example 3 (continued):

and hence:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$D = [\,0\,]$$

This matrix A is said to be in the "*companion form*"

that is because:

- the *elements above the main diagonal* are $= 1$;
- the last row contains the *characteristic equation coefficients* in the *inverse order* and with *opposite signals*;
- all other elements of the matrix are $= 0$.

In the general, matrix A in the "*companion form*" has the following aspect:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \left(\frac{-a_n}{a_o}\right) & \left(\frac{-a_{n-1}}{a_o}\right) & \left(\frac{-a_{n-2}}{a_o}\right) & \left(\frac{-a_{n-3}}{a_o}\right) & \cdots & \left(\frac{-a_1}{a_o}\right) \end{bmatrix}$$

where a_o, a_1, … , a_{n-1} and a_n are the *coefficients* of the *characteristic equation* $p(s)$:

$$p(s) = a_o s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

Note that matrices A of the 2 previous examples are also in the "*companion form*".

In the particular case, but very common, of $a_o = 1$, matrix A in the "*companion form*" has the following aspect:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}$$

where $a_1, \ldots, a_{n-1}$ and a_n are the *coefficients* of the *characteristic equation* $p(s)$:

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

If p = q = 1 (i.e., 1 input and 1 output) and m = *degree numerator of the transfer function* is *smaller than* *degree characteristic polynomial* (i.e., $m < n$), then we say that the system is in the "*companion form*" when

besides the matrix A being in the "*companion form*" we have matrices B, C and D in the forms:

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\ \beta_n \quad \beta_{n-1} \quad \dots \quad \beta_1\ ]$$

$$D = [\ 0\ ]$$

where $\beta_1, \dots, \beta_{n-1}$ e β_n, are the *coefficients* of the *transfer function numerator*, q(s):

$$q(s) = \beta_1 s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_{n-1} s + \beta_n$$

In the case of $p = q = 1$ (i.e., 1 input and 1 output) and m = *degree numerator of the transfer function* is equal to *degree characteristic polynomial* (i.e., $m = n$), then the *numerator of the transfer function* $q(s)$ is given by:

$$q(s) = \beta_o s^n + \beta_1 s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_{n-1} s + \beta_n$$

and we say that the system is in the "*companion form*" when besides the matrix A being in the "*companion form*", we have matrices B, C and D in the forms:

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C = [\, c_n \quad c_{n-1} \quad \dots \quad c_1 \,] \qquad D = [\, d_1 \,]$$

where $d_1 = \beta_o / a_o$

and $c_1, \dots, c_{n-1}$ e c_n are the *coefficients* of the *polynomial*, $r(s)$, the remainder of division $q(s)/p(s)$

$$r(s) = c_1 s^{n-1} + c_2 s^{n-2} + \dots + c_{n-1} s + c_n$$

Example 4:

If the *ordinary differential equation* (ODE) also had *derivatives* of u, the above choice would not be appropriate.

Consider the system described by:

$$y'' + 2y' + 2y = u' + 2u$$

Here the *transfer function* of the system is:

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^2+2s+2}$$

Example 4 (continued):

In this case we define the following *state variables*:

$$\begin{cases} X_1(s) = \dfrac{U(s)}{s^2 + 2s + 2} \\ X_2(s) = \dfrac{s \cdot U(s)}{s^2 + 2s + 2} \end{cases}$$

$$\begin{cases} s \cdot X_1(s) = X_2(s) \\ s^2 \cdot X_1(s) + 2s \cdot X_1(s) + 2 \cdot X_1(s) = U(s) \end{cases}$$

$sX_2(s)$ $X_2(s)$

Example 4 (continued):

thus:

$$\begin{cases} s \cdot X_1(s) = X_2(s) \\ s \cdot X_2(s) = -2 \cdot X_1(s) - 2 \cdot X_2(s) + U(s) \\ Y(s) = 2 \cdot X_1(s) + X_2(s) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

Example 4 (continued):

and hence:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [\,2 \quad 1\,] \qquad D = [\,0\,]$$

Note that *matrix* A of this example is in the *companion form* again, since the *characteristic equation* of the system is:

$$p(s) = s^2 + 2s + 2$$

Example 5:

Consider the system which *transfer function* is given by:

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 7s + 3}{s^2 + 4s - 2}$$

In this case we have a *second order* system and thus, it has 2 poles

But since the numerator of the *transfer function* has the same degree as the denominator, the system also has 2 zeros

Example 5 (continued):

Firstly, dividing the *numerator* by the *denominator*:

$$\begin{array}{r|l} 2s^2 + 7s + 3 & s^2 + 4s - 2 \\ \hline -2s^2 - 8s + 4 & 2 \\ \hline -s + 7 & \end{array}$$

We got the *quotient* 2 e o *remainder* $(-s+7)$. Then,

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 7s + 3}{s^2 + 4s - 2} = 2 + \frac{-s + 7}{s^2 + 4s - 2}$$

Example 5 (continued):

that is,

$$Y(s) = \frac{(-s+7)\cdot U(s)}{s^2+4s-2} + 2\cdot U(s)$$

Now defining the *state variable*

$$\begin{cases} X_1(s) = \dfrac{U(s)}{s^2+4s-2} \\ X_2(s) = \dfrac{s\cdot U(s)}{s^2+4s-2} \end{cases}$$

Example 5 (continued):

we have that:

$$\begin{cases} s \cdot X_1(s) = \dfrac{s \cdot U(s)}{s^2 + 4s - 2} = X_2(s) \\ s^2 \cdot X_1(s) + 4\, s \cdot X_1(s) - 2\, X_1(s) = U(s) \end{cases}$$

$s^2 \cdot X_1(s) \rightarrow s \cdot X_2(s)$ $\qquad$ $s \cdot X_1(s) \rightarrow X_2(s)$

thus:

$$\Rightarrow \begin{cases} s \cdot X_1(s) = X_2(s) \\ s \cdot X_2(s) = 2X_1(s) - 4X_2(s) + U(s) \end{cases}$$

Example 5 (continued):

and observe that the *output* $y(t)$:

$$Y(s) = \frac{(-s+7)\cdot U(s)}{s^2+4s-2} + 2\cdot U(s)$$

can be rewritten as:

$$Y(s) = 7\cdot\frac{U(s)}{2s^2+4s-2} - \frac{s\cdot U(s)}{2s^2+4s-2} + 2\cdot U(s)$$

$X_1(s)$ $X_2(s)$

$$Y(s) = 7\cdot X_1(s) - X_2(s) + 2\cdot U(s)$$

Example 5 (continued):

thus:

$$\begin{cases} s \cdot X_1(s) = X_2(s) \\ s \cdot X_2(s) = 2 \cdot X_1(s) - 4 \cdot X_2(s) + U(s) \\ Y(s) = 7 \cdot X_1(s) - X_2(s) + 2 \cdot U(s) \end{cases}$$

and therefore we have:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 2x_1 - 4x_2 + u \\ y = 7x_1 - x_2 + 2u \end{cases}$$

Example 5 (continued):

then:

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 7 & -1 \end{bmatrix}}_{C} x + \underbrace{2}_{D} u \end{cases}$$

and therefore:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = [\,7 \;\; -1\,] \qquad D = [\,2\,]$$

Observe that a matrix A here in this example is also in the *companion form*

the *characteristic equation* and
the *poles* of the system

The *characteristic equation* and the *poles* of the system

A system described in the form of *state equations*

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

has its *characteristic polynomial* given by:

$$p(s) = \det\{[\,sI - A\,]\}$$

The *poles* of the system are the "*eigenvalues*" of A, which can be *repeated*, i.e., *double*, *triple*, etc.

It is well known that the *eigenvalues* of A are the *roots* of the *characteristic polynomial*

$$p(s) = \det [\, s \cdot I - A \,]$$

Example 6:

For system of *example 1* the matrix A is given by:

$$A = \begin{bmatrix} 0 & 1 \\ \left(-\frac{k}{m}\right) & \left(-\frac{\mu}{m}\right) \end{bmatrix}$$

thus, the *characteristic polynomial* $p(s) = \det[\, s \cdot I - A \,]$

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ (k/m) & (s + \mu/m) \end{bmatrix}$$

and therefore:

$$p(s) = s^2 + \frac{\mu}{m} s + \frac{k}{m}$$

Example 7:

For the system of *example 2* the matrix A is given by:

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

thus, the *characteristic polynomial* $p(s) = \det[\, s \cdot I - A \,]$

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ 3 & (s+4) \end{bmatrix}$$

and hence:

$$p(s) = s^2 + 4s + 3$$

and the *poles* of the system are the roots of $p(s)$:

$$s = -1 \quad \text{e} \quad s = -3$$

Example 8:

For the system of *example 3* the matrix A is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix}$$

thus, the *characteristic polynomial* $p(s) = \det[\, s \cdot I - A \,]$

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 5 & (s+4) \end{bmatrix}$$

Example 8 (continued):

and therefore:

$$p(s) = s^3 + 4s^2 + 5s$$

and the *poles* of the system are the roots of $p(s)$:

$$s = 0, \qquad s = -2 + j \qquad \text{e} \qquad s = -2 - j$$

Example 9:

For the system of the *example 4*, the matrix A is given by:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

thus, the *characteristic polynomial* $p(s) = \det[\, s \cdot I - A \,]$

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ 2 & (s+2) \end{bmatrix}$$

and hence:

$$p(s) = s^2 + 2s + 2$$

and the *poles* of the system are the roots of $p(s)$:

$$s = -1 + j \quad \text{e} \quad s = -1 - j$$

Example 10:

For the system of the *example 5*, the matrix A is given by:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$$

thus, the *characteristic polynomial* $p(s) = \det[\, s \cdot I - A \,]$

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ -2 & (s+4) \end{bmatrix}$$

and hence:

$$p(s) = s^2 + 4s - 2$$

and the *poles* of the system are the roots of $p(s)$:

$$s = 0{,}45 \quad \text{e} \quad s = -4{,}45$$

equivalent representations

Equivalent representations

Consider a system described by *state equations*

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

which the *state variable* is $x(t)$.

Now defining a new *state variable* $\bar{x}$ as:

$$\bar{x} = P\,x$$

P being invertible.

thus, since:

$$\dot{\bar{x}} = P\,\dot{x}$$

we have that:

$$\begin{cases} x = P^{-1}\,\bar{x} \\ \dot{x} = P^{-1}\,\dot{\bar{x}} \end{cases}$$

and substituting the *state equations* we get:

$$\begin{cases} P^{-1}\dot{\bar{x}} = AP^{-1}\bar{x} + Bu \\ y = CP^{-1}\bar{x} + Du \end{cases}$$

$$\begin{cases} \dot{\bar{x}} = \overbrace{PAP^{-1}}^{\bar{A}}\bar{x} + \overbrace{PB}^{\bar{B}}u \\ y = \underbrace{CP^{-1}}_{\bar{C}}\bar{x} + \underbrace{D}_{\bar{D}}u \end{cases}$$

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{cases}$$

State Equations

that is:

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,u \\ y = \bar{C}\,\bar{x} + \bar{D}\,u \end{cases}$$

this is another *representation* of the same system in *state equations*

where:

$$\begin{cases} \bar{A} = P\,A\,P^{-1} \\ \bar{B} = P\,B \\ \bar{C} = C\,P^{-1} \\ \bar{D} = D \end{cases}$$

Note that the *input* u and the *output* y do not change.

Only the internal representation of the system (as *state variable*)

Example 11:

Consider a system of the 2nd order of Example 4, which state equations are:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 2 & 1 \end{bmatrix} x \end{cases}$$

The original state variable is:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

By choosing

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 11 (continued):

we have that

$$\bar{x}(t) = Px = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}$$

that is,
the new *state variable* $\bar{x}$ is the old *state variable* x with its component swapped

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$$

$$\bar{B} = PB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{D} = D = 0$$

$$\bar{C} = CP^{-1} = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Example 11 (continued):

$$\begin{cases} \dot{\bar{x}} = \overbrace{\begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}}^{\bar{A}} \bar{x} + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\bar{B}} u \\ y = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{\bar{C}} \bar{x} \end{cases}$$

Note that *matrix* P of this example is its own inverse:

$$P = P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note also that:

$$P = P^{-1} \quad \Rightarrow \quad P \cdot P^{-1} = P \cdot P = P^2$$

but $P \cdot P^{-1} = I$, thus,

$$P^2 = I$$

These matrices are called idempotent.

Example 12: Now consider the *system of the 3rd order* of Example 3 above:

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

For the new *state variable* $\bar{x}$ be the same as the old x, only changing the third component x_3 by the double: $\bar{x}_3 = 2\, x_3$, the choice of P should be:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example 12 (continued):

and then we have that

$$\overline{x}(t) = P\,x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ 2\,x_3(t) \end{bmatrix}$$

$$\overline{A} = PAP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0{,}5 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0{,}5 \\ 0 & -10 & -4 \end{bmatrix}$$

$$\overline{B} = PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$\overline{C} = CP^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0{,}5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\overline{D} = D = 0$$

Example 12 (continued):

thus, the *state equations* below are a different representation of the same system

$$\begin{cases} \dot{\bar{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0,5 \\ 0 & -10 & -4 \end{bmatrix}}_{\bar{A}} \bar{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}}_{\bar{B}} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\bar{C}} \bar{x} \end{cases}$$

$\bar{D} = 0$

conversion from the state equation

to

transfer function

Conversion from the State Equation to Transfer Function

In order to convert the representation of a system in *state equations*

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

to *transfer function*, the expression is given by,

$$\frac{Y(s)}{U(s)} = C \cdot (sI - A)^{-1} \cdot B + D$$

Example 13:

Consider the second order system of example 4 given by its *state equations*

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

To calculate the *transfer function*, first we find the matrix $(s\,I - A)$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}$$

Example 13 (continued):

and its inverse $(sI - A)^{-1}$

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+2}{s^2+2s+2} & \dfrac{1}{s^2+2s+2} \\ \dfrac{-2}{s^2+2s+2} & \dfrac{s}{s^2+2s+2} \end{bmatrix}$$

and hence, as $D = 0$ in this case, $T.F. = C(sI - A)^{-1}B$

$$\frac{Y(s)}{U(s)} = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dfrac{s+2}{s^2+2s+2} & \dfrac{1}{s^2+2s+2} \\ \dfrac{-2}{s^2+2s+2} & \dfrac{s}{s^2+2s+2} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

Example 13 (continued):

thus, the *transfer function* of system is given by:

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^2+2s+2}$$

which agrees with example 4.

Note that in order to find the *characteristic equation* only, it would be enough to calculate:

$$p(s) = \det[\, s \cdot I - A \,] =$$

$$= s^2 + 2s + 2$$

as we have seen in example 9.

Example 14:

Consider the *second order* system of the example 5 given by the *state equation*

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 7 & -1 \end{bmatrix}}_{C} x + \underbrace{[2]}_{D} u \end{cases}$$

To calculate the *transfer function*, first we find the matrix $(s\,I - A)$

$$(sI - A) = \begin{bmatrix} s & -1 \\ -2 & s+4 \end{bmatrix}$$

Example 14 (continued):

and its inverse $(s\,I - A)^{-1}$

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+4}{s^2+4s-2} & \dfrac{1}{s^2+4s-2} \\ \dfrac{2}{s^2+4s-2} & \dfrac{s}{s^2+4s-2} \end{bmatrix}$$

and hence, the *transfer function*

$$\frac{Y(s)}{R(s)} = \underbrace{[7 \quad -1]}_{C} \cdot \underbrace{\begin{bmatrix} \dfrac{s+4}{s^2+4s-2} & \dfrac{1}{s^2+4s-2} \\ \dfrac{2}{s^2+4s-2} & \dfrac{s}{s^2+4s-2} \end{bmatrix}}_{(sI-A)^{-1}} \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} + \underbrace{2}_{D}$$

Example 14 (continued):

thus, the *transfer function* of the system is given by:

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 7s + 3}{s^2 + 4s - 2}$$

which agrees with Example 5.

Note that in order to find the *characteristic equation* only, it would be enough to calculate:

$$p(s) = \det[s \cdot I - A] =$$

$$= s^2 + 4s - 2$$

as we have seen in example 10.

Example 15:

Consider the *third order* system of the example 3 given by the *state equation*

$$\begin{cases} \dot{x} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}^{B} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x \end{cases}$$

$D = 0$

To calculate the *transfer function*, first we find the matrix $(s\,I - A)$

$$(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 5 & s+4 \end{bmatrix}$$

Example 15 (continued):

and its inverse $(sI - A)^{-1}$

$$(sI - A)^{-1} =$$

$$= \begin{bmatrix} \dfrac{s^2 + 4s + 5}{s^3 + 4s^2 + 5s} & \dfrac{s + 4}{s^3 + 4s^2 + 5s} & \dfrac{1}{s^3 + 4s^2 + 5s} \\ 0 & \dfrac{s^2 + 4s}{s^3 + 4s^2 + 5s} & \dfrac{s}{s^3 + 4s^2 + 5s} \\ 0 & \dfrac{-5s}{s^3 + 4s^2 + 5s} & \dfrac{s^2}{s^3 + 4s^2 + 5s} \end{bmatrix}$$

Example 15 (continued):

hence, the *transfer function* T.F. = $C\,(s\,I - A)^{-1}\,B$

$$\frac{Y(s)}{R(s)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \cdot \underbrace{(s\,I - A)^{-1}}_{(s\,I - A)^{-1}} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{B}$$

thus, the *transfer function* of the system is given by:

$$\frac{Y(s)}{R(s)} = \frac{3}{s^3 + 4s^2 + 5s}$$

which agrees with example 3.

Example 15 (continued):

Note that in order to find the *characteristic equation* only, it would be enough to calculate:

$$p(s) = \det [\, s \cdot I - A \,]$$

$$= s^3 + 4s^2 + 5s$$

$$= s\,(s^2 + 4s + 5)$$

as we have seen in example 8.

to be continued

(*next class*)

part II

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Thank you!
Obrigado!

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