Control Systems

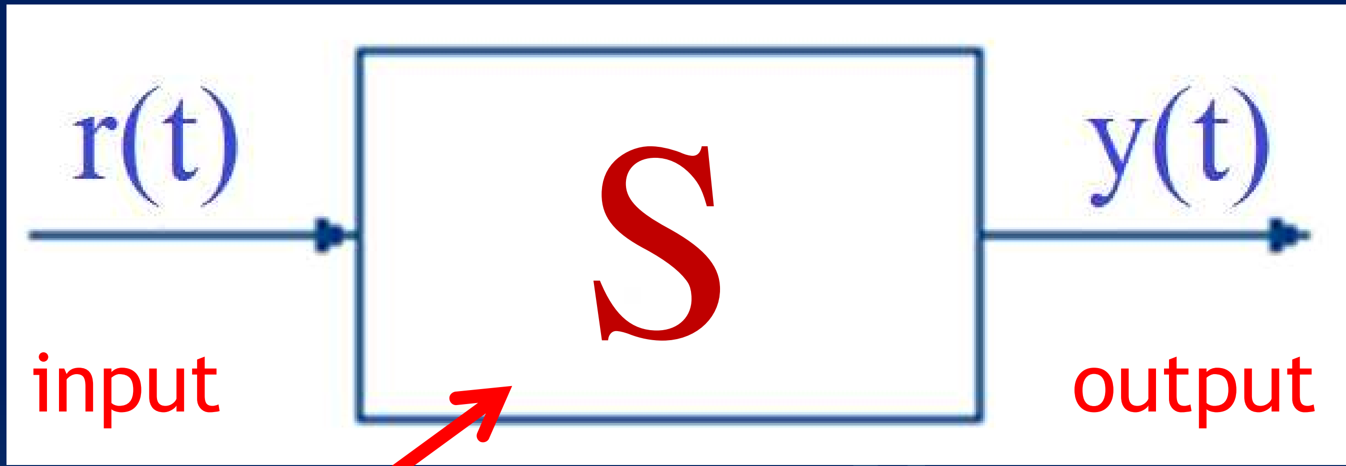
7

"Time domain analysis"

part II – 2nd order systems

J. A. M. Felippe de Souza

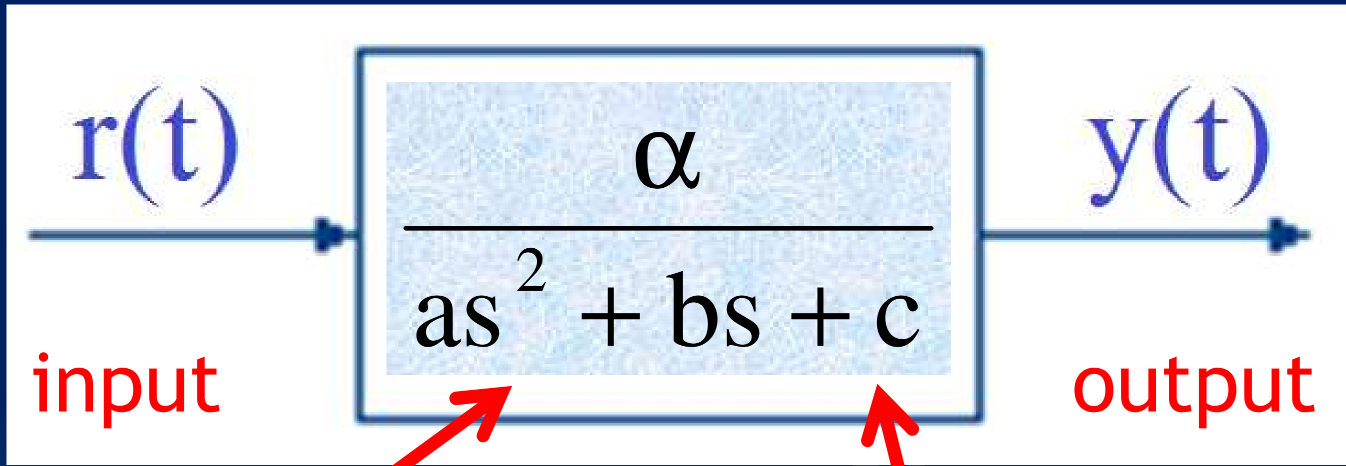
Time domain analysis - 2nd order systems



Second order systems

of the type

$$G(s) = \frac{\alpha}{as^2 + bs + c}$$

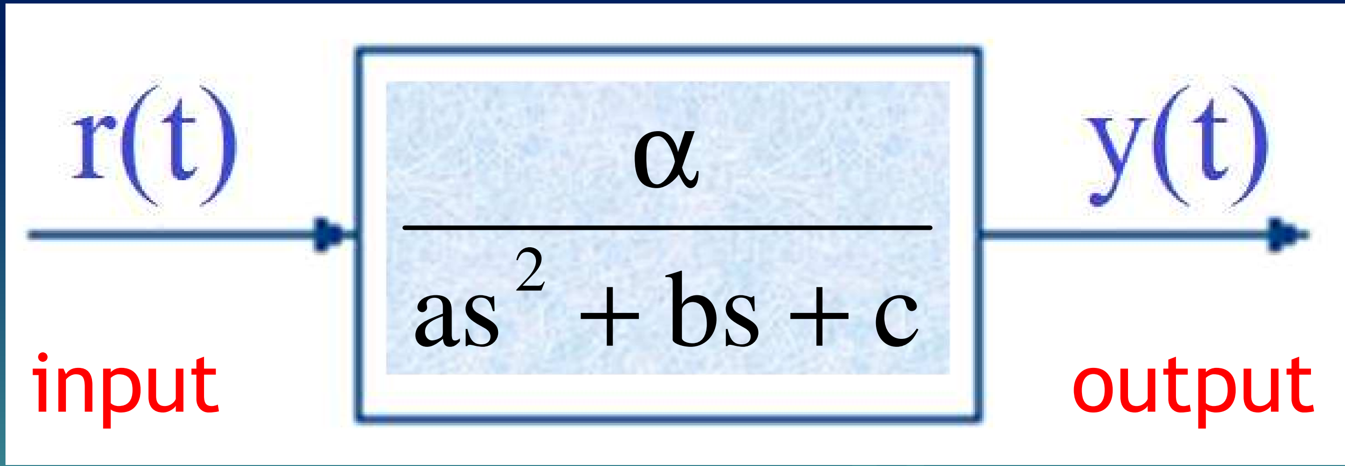


Second order systems

of the type

$$G(s) = \frac{\alpha}{as^2 + bs + c}$$

Second order systems:



that is:

$$\frac{Y(s)}{R(s)} = \frac{\alpha}{as^2 + bs + c}$$

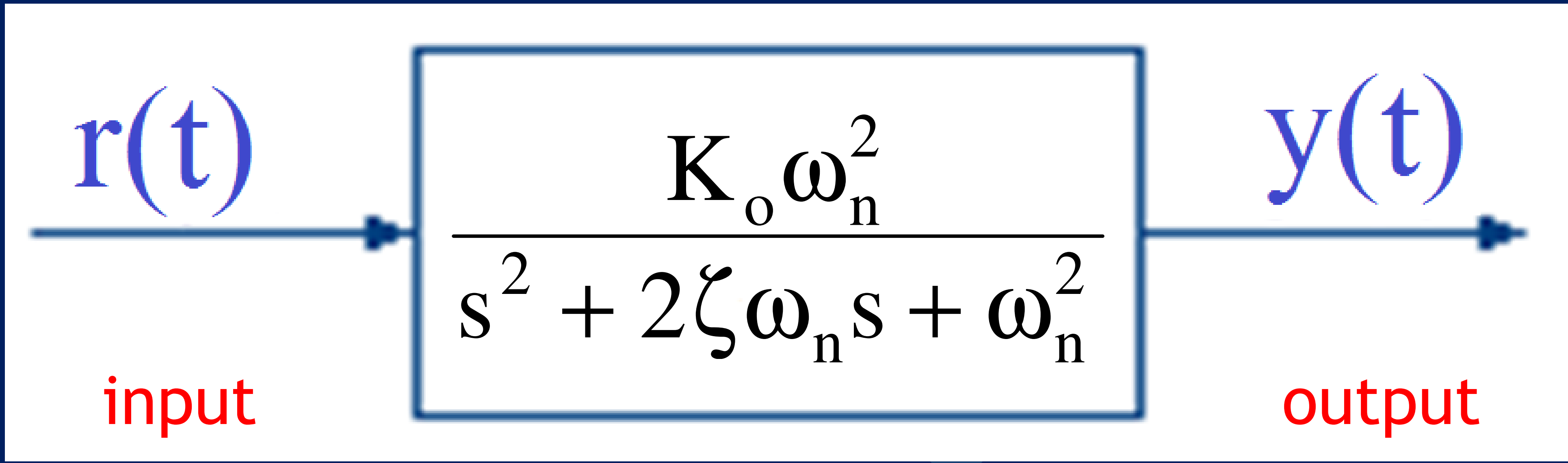
$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{\dfrac{\alpha}{a}}{s^2 + \dfrac{b}{a}s + \dfrac{c}{a}}$$

$\dfrac{\alpha}{a} \rightarrow K_o\omega_n^2$

$\dfrac{b}{a} \rightarrow 2\zeta\omega_n$

$\dfrac{c}{a} \rightarrow \omega_n^2$

Time domain analysis - 2nd order systems



K_o = gain of the system

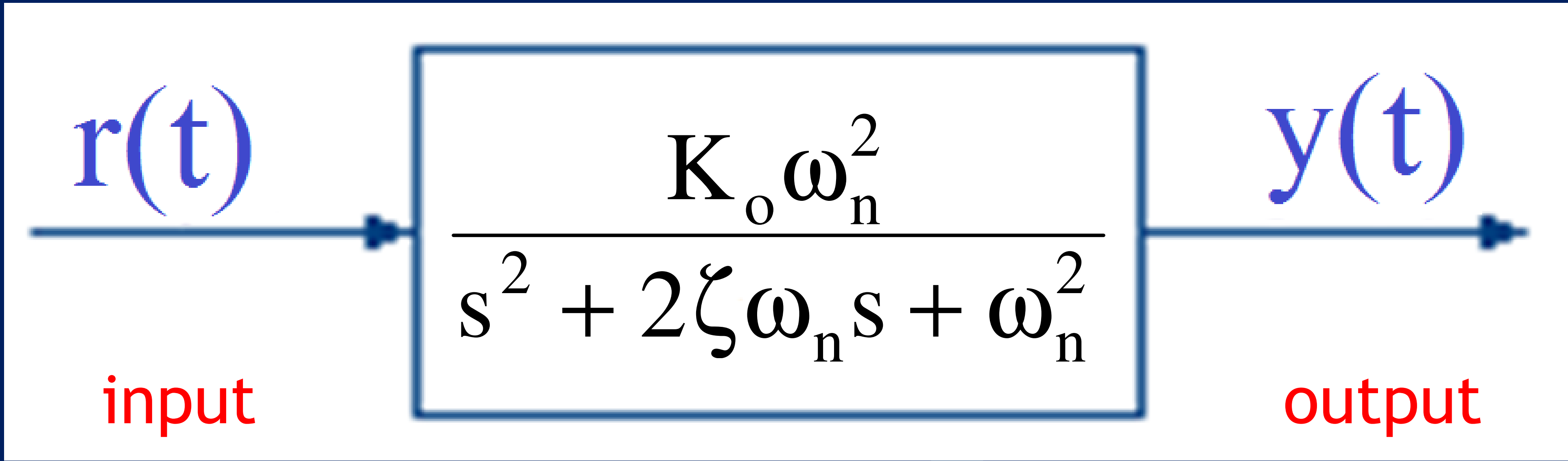
ζ = damping coefficient

ω_n = natural frequency

the *transfer function* can be rewritten as:

$$\frac{Y(s)}{R(s)} = \frac{K_o\,\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Time domain analysis - 2nd order systems



K_o = gain of the system

ζ = damping coefficient

ω_n = natural frequency

Besides these 3 above parameters, we also have

ω_d = damping frequency

$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2} \qquad 0 < \zeta \leq 1$$

Example 1:

$$\frac{Y(s)}{R(s)} = \frac{3}{4s^2 + 12s + 1}$$

poles:
s = -2,914
s = -0,086

poles are real and distinct

$K_o = 3$ $\zeta = 3$ $\omega_n = 0{,}5$

Example 2:

$$\frac{Y(s)}{R(s)} = \frac{3}{s^2 + 2s + 1}$$

poles:
s = -1
(duple)

$K_o = 3$ $\zeta = 1$ $\omega_n = 1$ $\omega_d = 0$

poles are real and repeated

Example 3:

$$\frac{Y(s)}{R(s)} = \frac{3}{2s^2 + 2s + 2}$$

poles:
$s = -0{,}5 \pm 0{,}866j$

complex conjugate poles

$K_o = 1{,}5$ $\zeta = 0{,}5$ $\omega_n = 1$ $\omega_d = 0{,}866$

Example 4:

$$\frac{Y(s)}{R(s)} = \frac{3}{s^2 + 1}$$

poles:
$s = \pm j$
(pure imaginary)

complex conjugate poles

$K_o = 3$ $\zeta = 0$ $\omega_n = \omega_d = 1$

Characteristic equation:

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\Delta = 4\,\zeta^2\,\omega_n^2 - 4\,\omega_n^2 =$$
$$= 4\,\omega_n^2\,(\zeta^2 - 1)$$

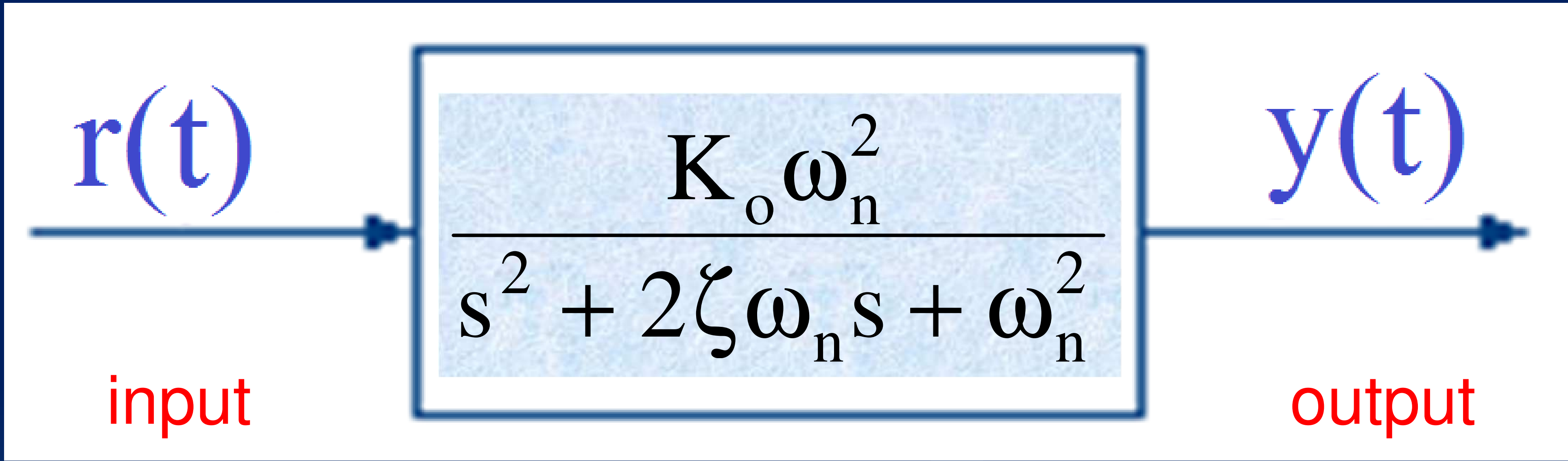
$$\begin{cases} \Delta > 0 \;\rightarrow\; (\zeta^2 - 1) > 0 \;\rightarrow\; \zeta^2 > 1 \rightarrow \zeta > 1 \\ \Delta = 0 \;\rightarrow\; (\zeta^2 - 1) = 0 \;\rightarrow\; \zeta^2 = 1 \rightarrow \zeta = 1 \\ \Delta < 0 \;\rightarrow\; (\zeta^2 - 1) < 0 \;\rightarrow\; \zeta^2 < 1 \rightarrow \zeta < 1 \end{cases}$$

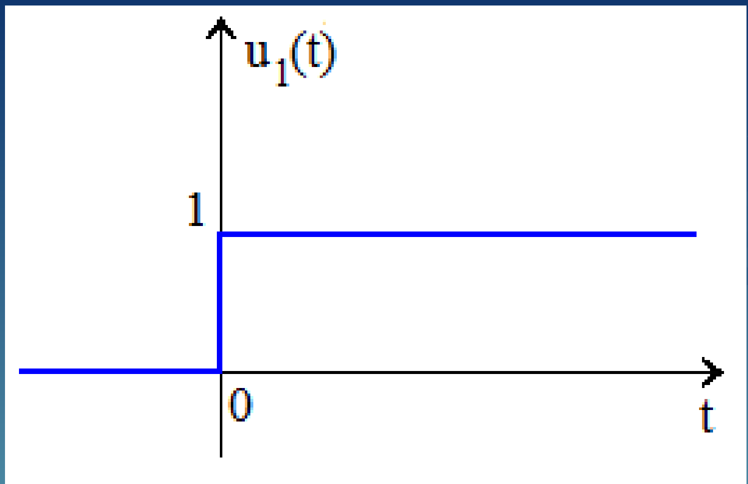
$\zeta > 1 \rightarrow$ *poles are real and distinct*

$\zeta = 1 \rightarrow$ *poles are real and repeated*

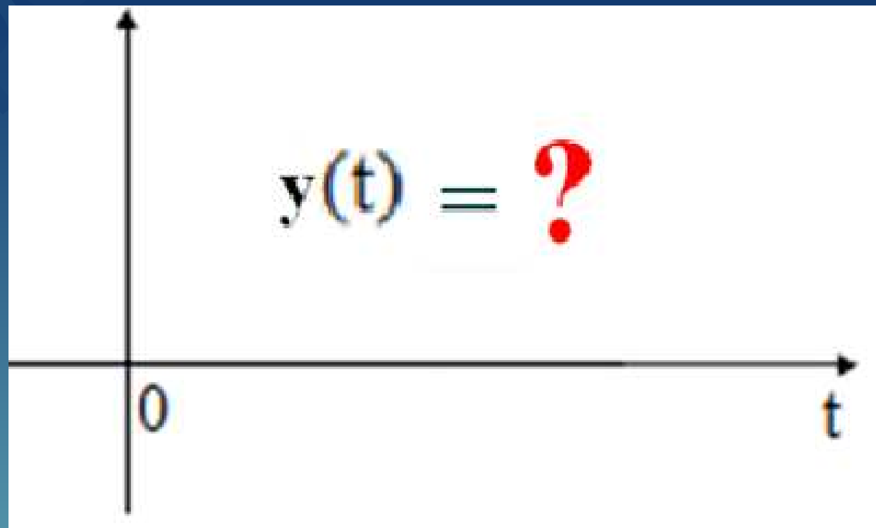
$0 < \zeta < 1 \rightarrow$ *poles are complex conjugate*

Time domain analysis - 2nd order systems



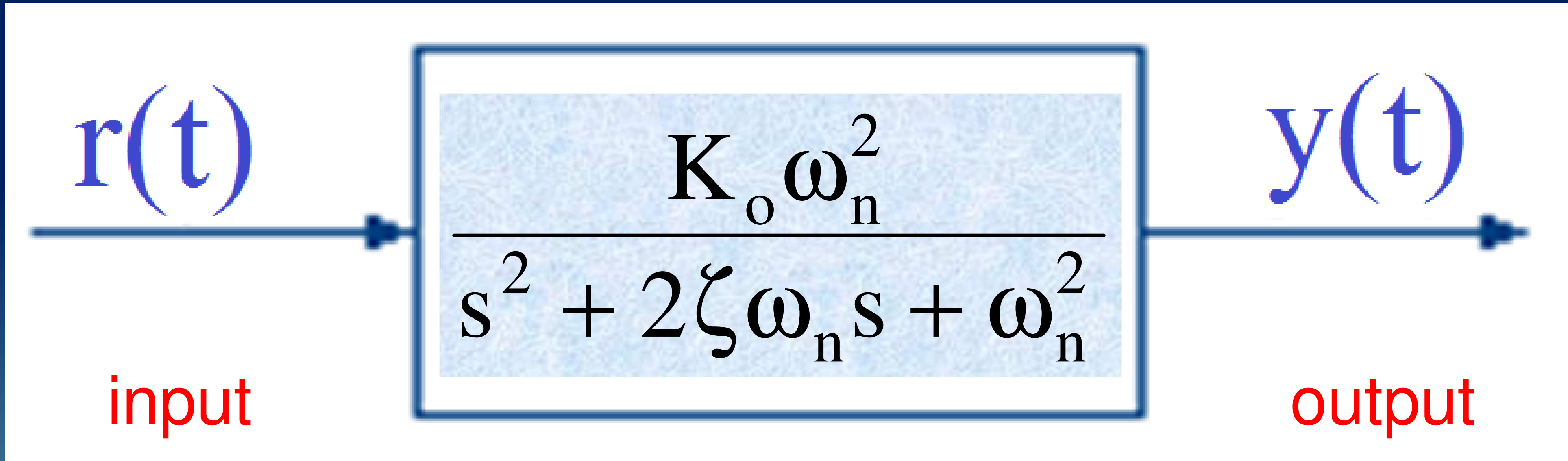


unit step input



What is the output?
(*step response*)

Time domain analysis - 2nd order systems

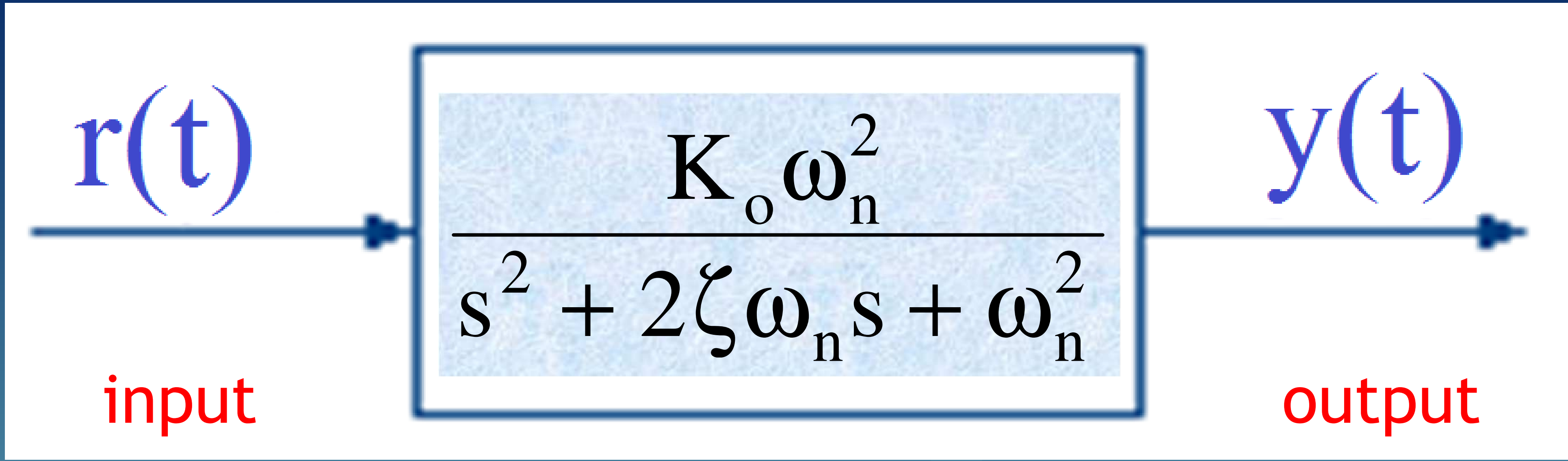


$$Y(s) = \frac{K_o \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot R(s)$$

and since r(t) = unit step:

$$Y(s) = \frac{K_o \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

Time domain analysis - 2nd order systems



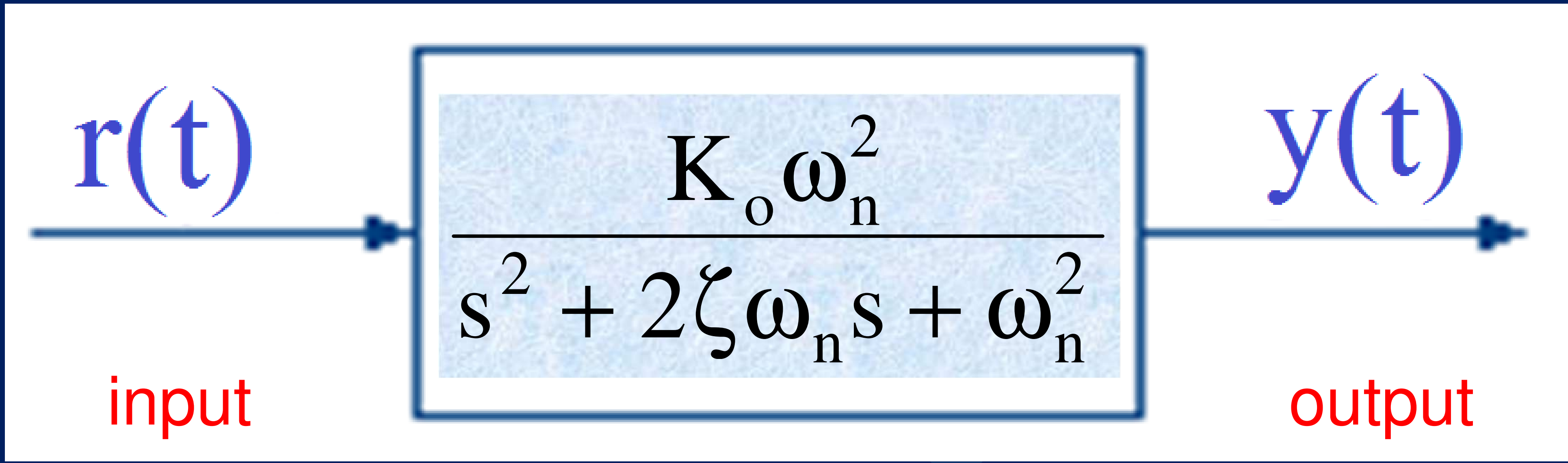
$$y(t) = \mathscr{L}^{-1}[Y(s)]$$

the unit step response depends on the value of ζ:

a) $0 < \zeta < 1$ (*under damping*)

b) $\zeta = 1$ (*critical damping*)

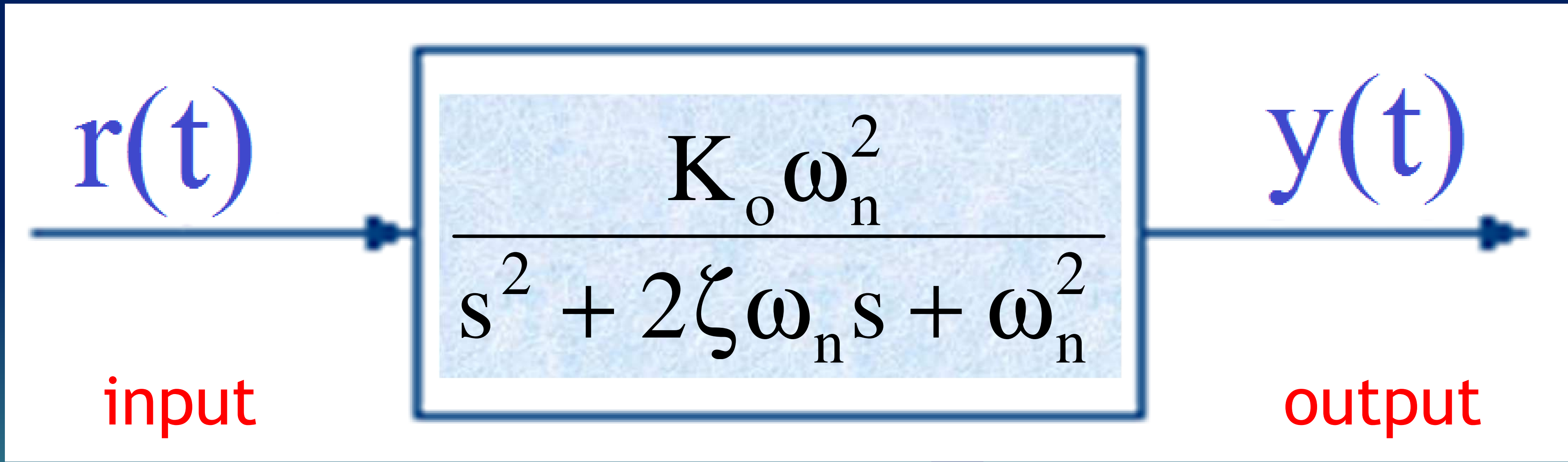
c) $\zeta > 1$ (*over damping*)



Let us now see $y(t)$, the outputs to the unit step in these 3 cases, starting with case (a)

a) $0 < \zeta < 1$ (*under damping*)

Time domain analysis - 2nd order systems



$$y(t) = \mathscr{L}^{-1}[Y(s)]$$

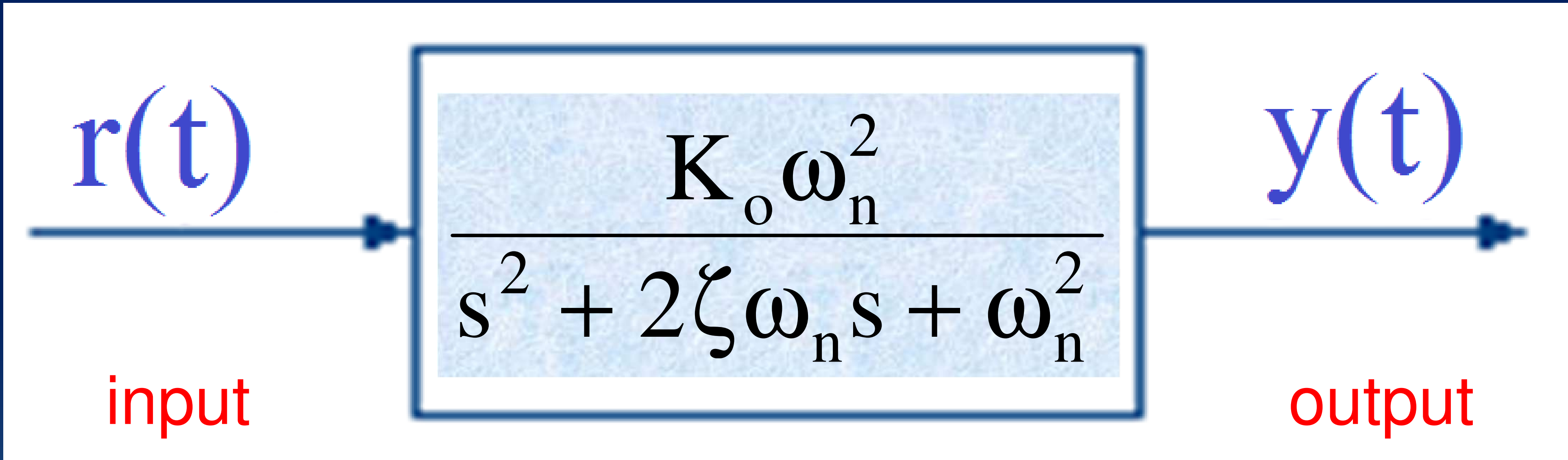
So, in the case $0 < \zeta < 1$ (*under damping*) the unit step response is:

$$y(t) = K_o\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot \operatorname{sen}\omega_d t\right)\right], \quad t > 0$$

where

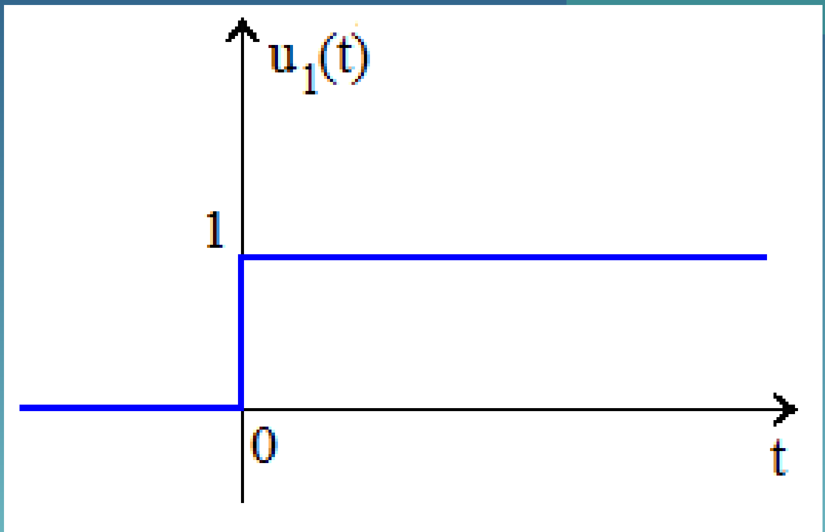
$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$$ (*damping frequency*)

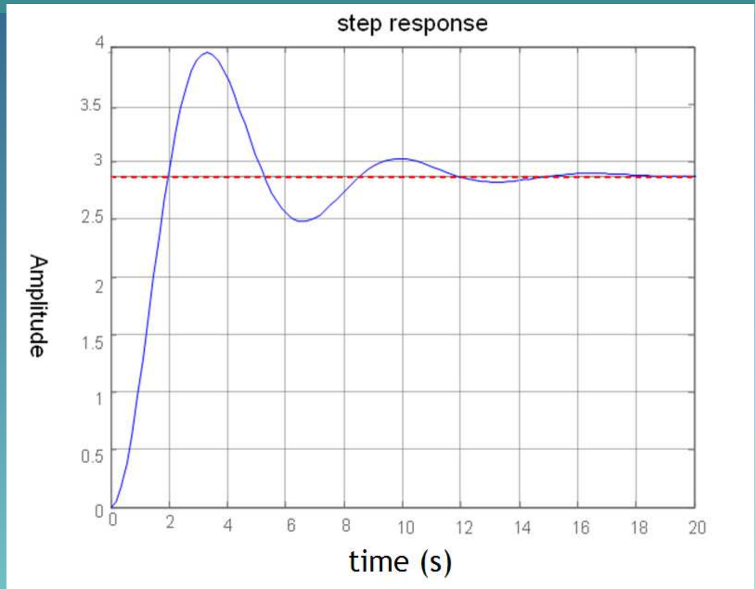
Time domain analysis - 2nd order systems



unit step response :

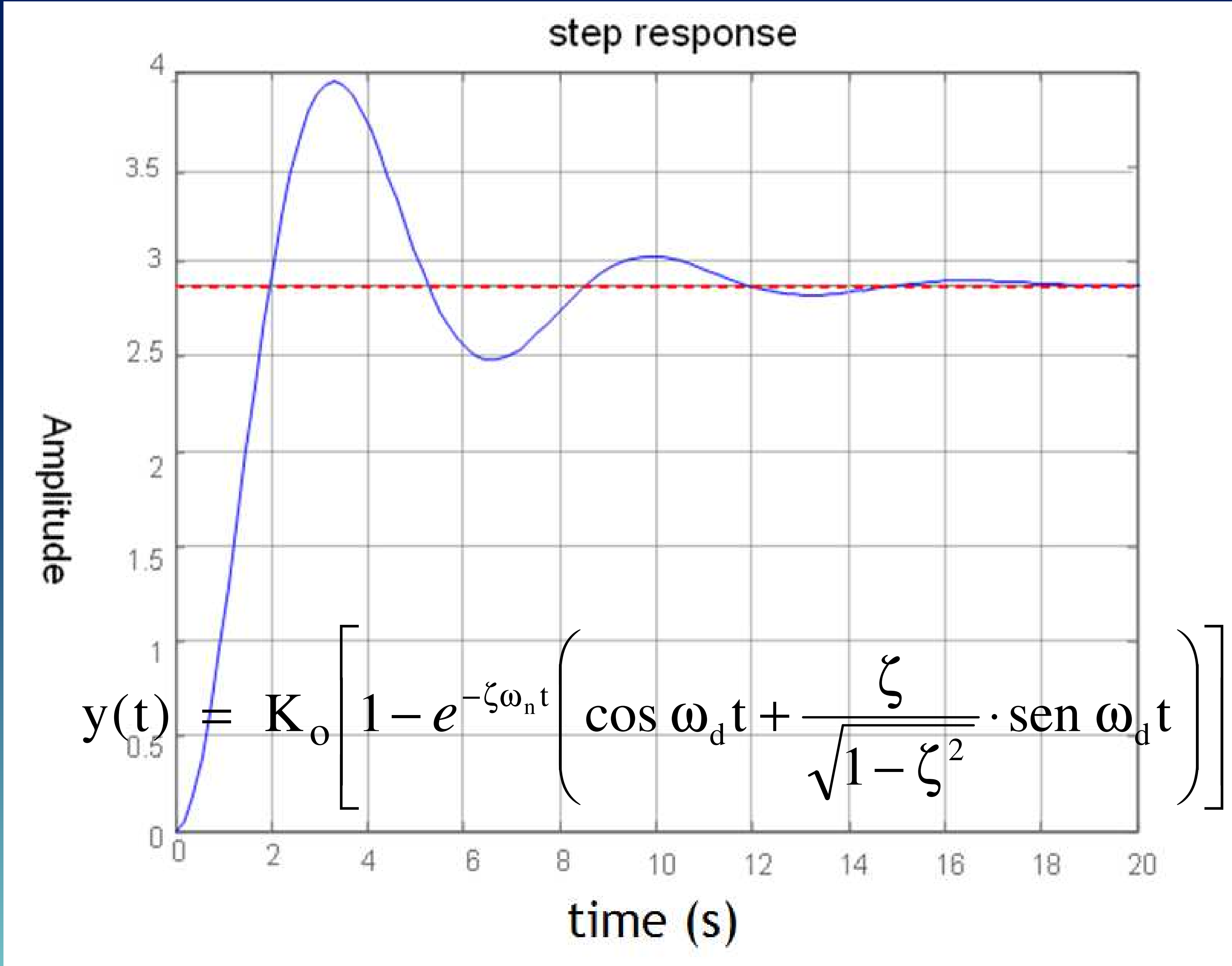
$$y(t) = K_o\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot \text{sen}\,\omega_d t\right)\right]$$



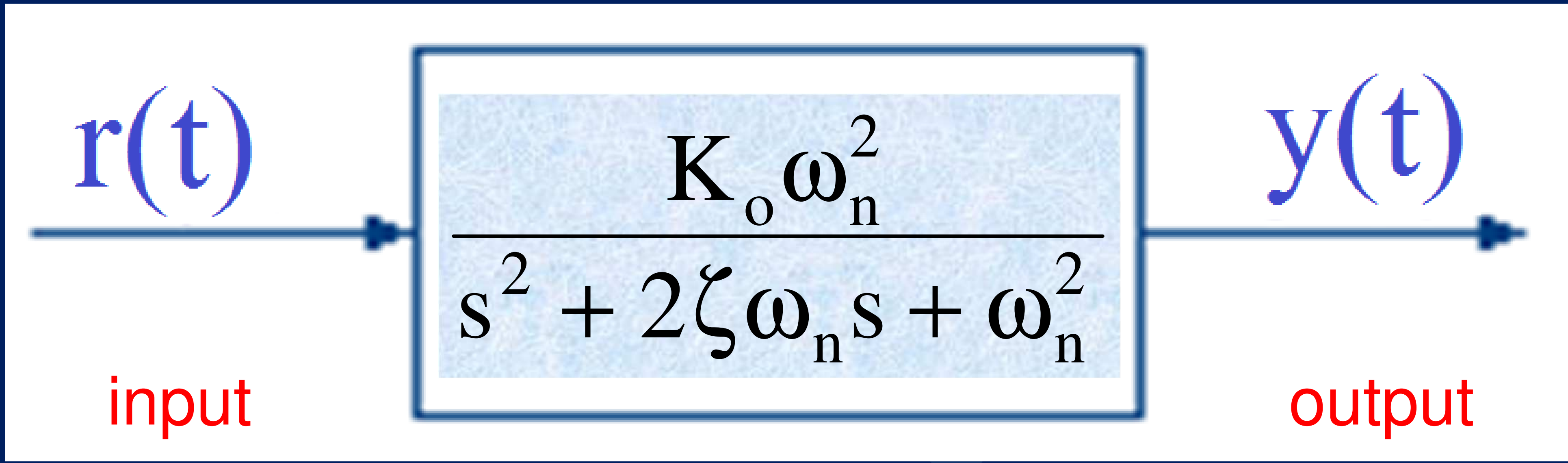


Time domain analysis - 2nd order systems

unit step response:

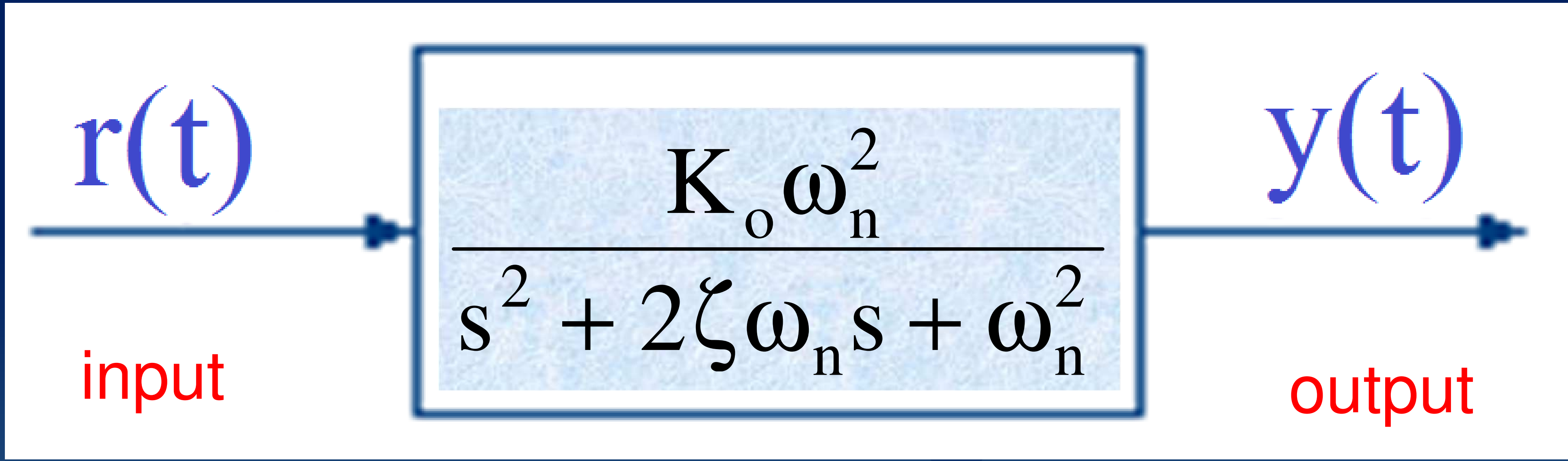


$$y(t) = K_o\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot \mathrm{sen}\,\omega_d t\right)\right]$$



Let us now see $y(t)$, the output to the unit step for case (b)

b) $\zeta = 1$ (*critical damping*)

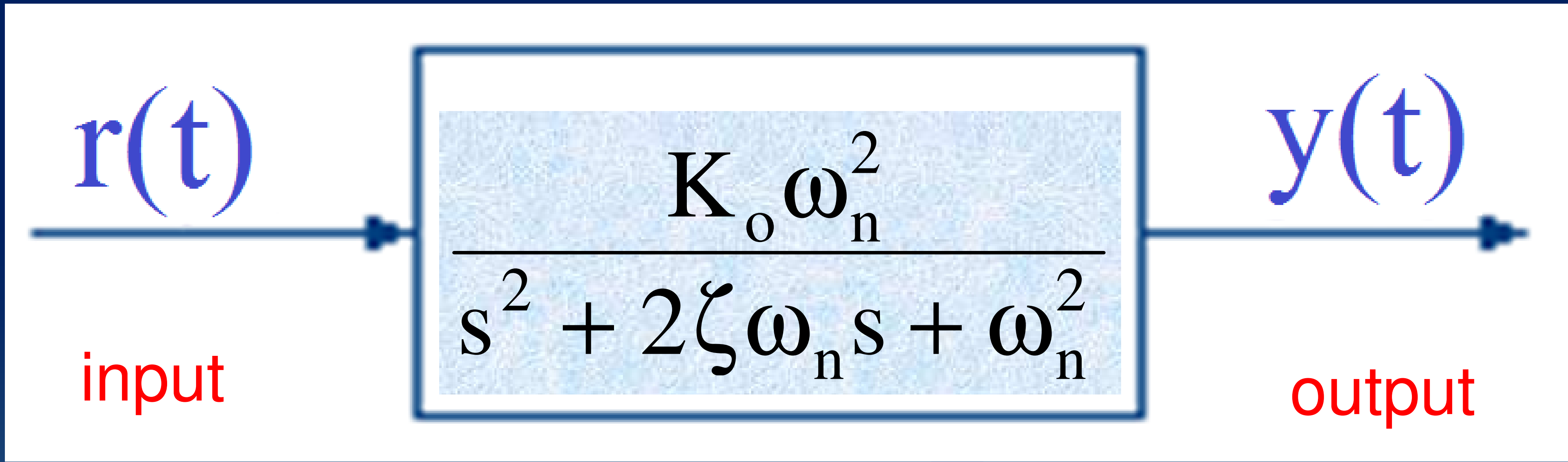


$$y(t) = \mathscr{L}^{-1}\left[Y(s)\right]$$

In the case $\zeta = 1$ (*critical damping*) the unit step response is:

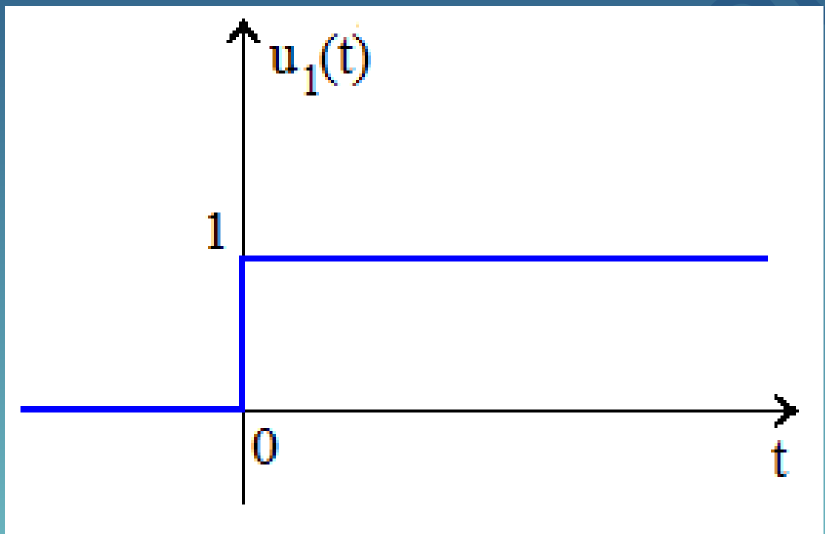
$$y(t) = K_o\left[1 - e^{-\zeta\omega_n t}\cdot\left(1 + \omega_n t\right)\right], \qquad t > 0$$

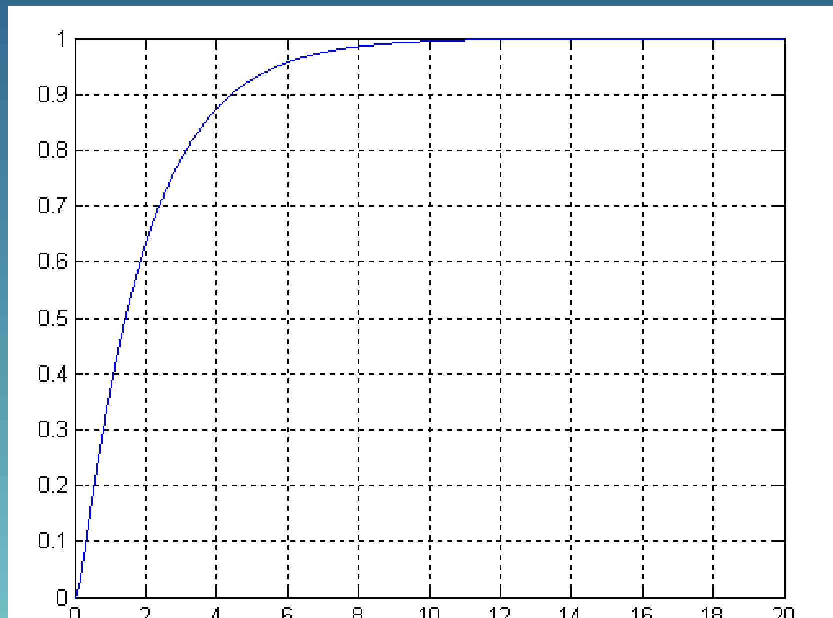
Time domain analysis - 2nd order systems



unit step response :

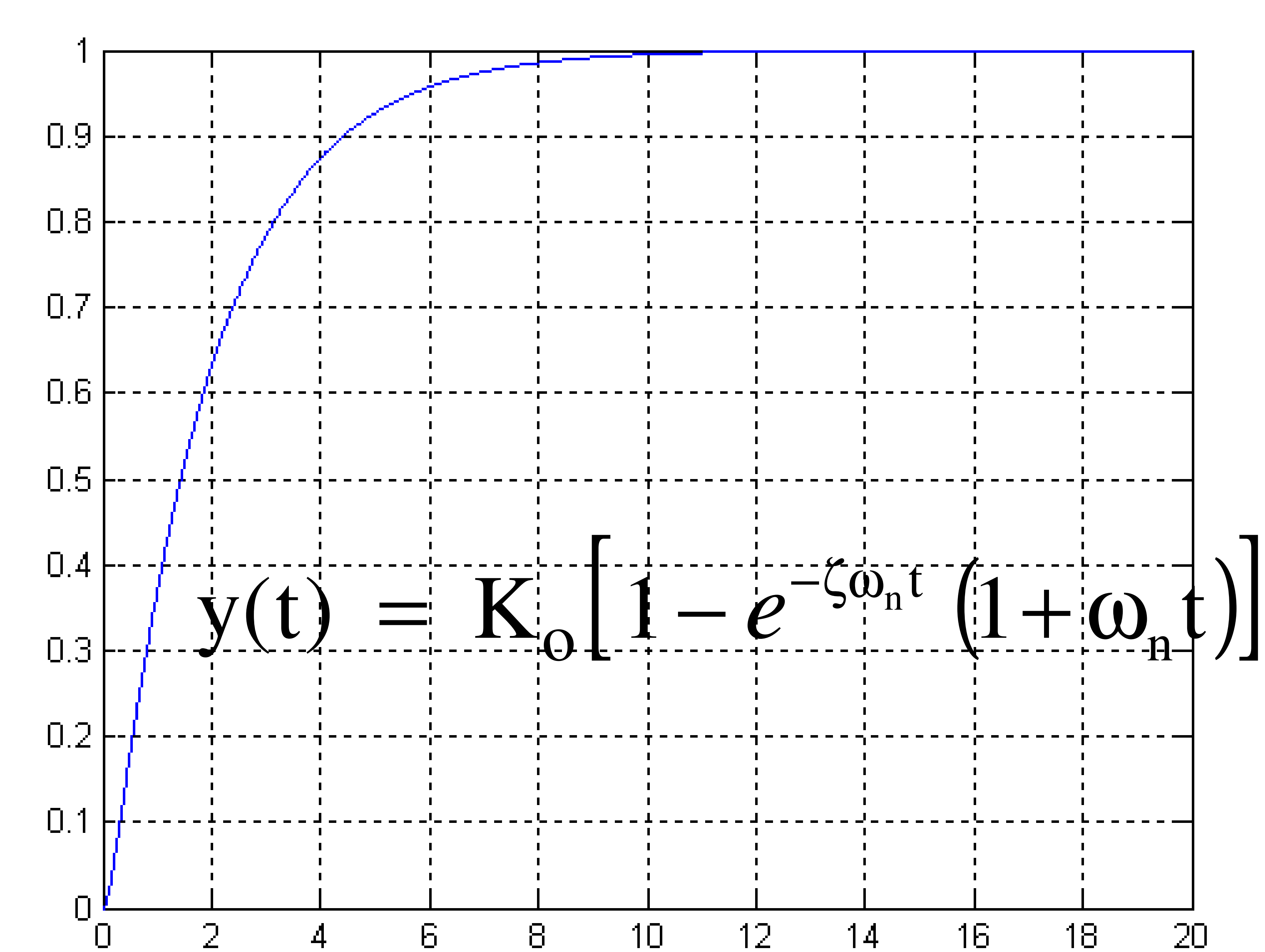
$$y(t) = K_o\left[1 - e^{-\zeta\omega_n t}\left(1 + \omega_n t\right)\right]$$



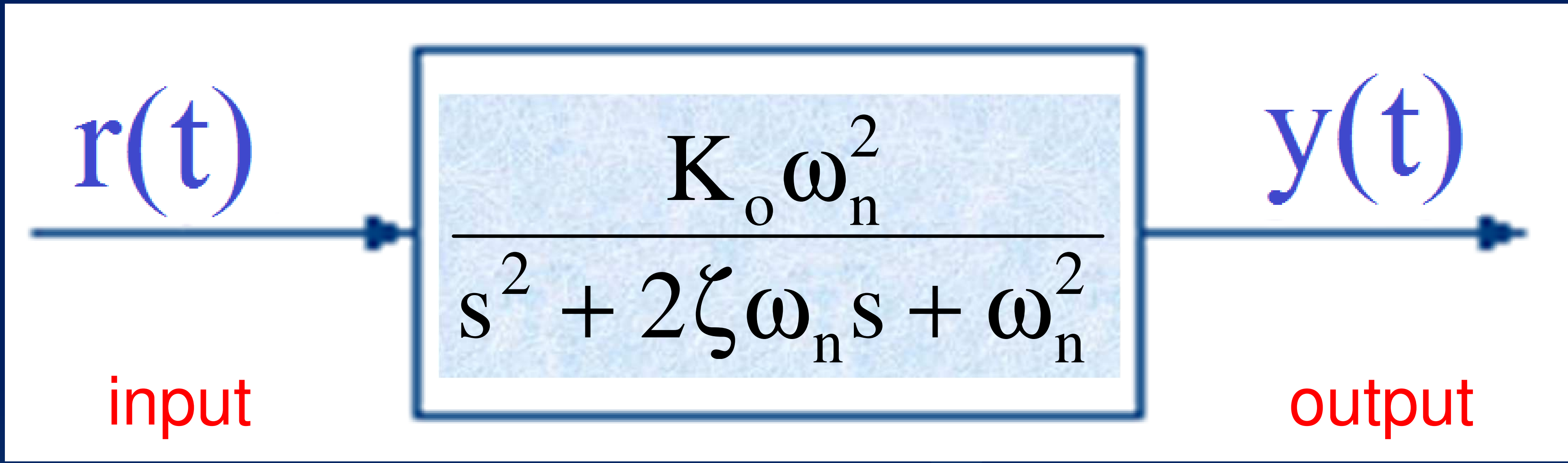


Time domain analysis - 2nd order systems

unit step response :



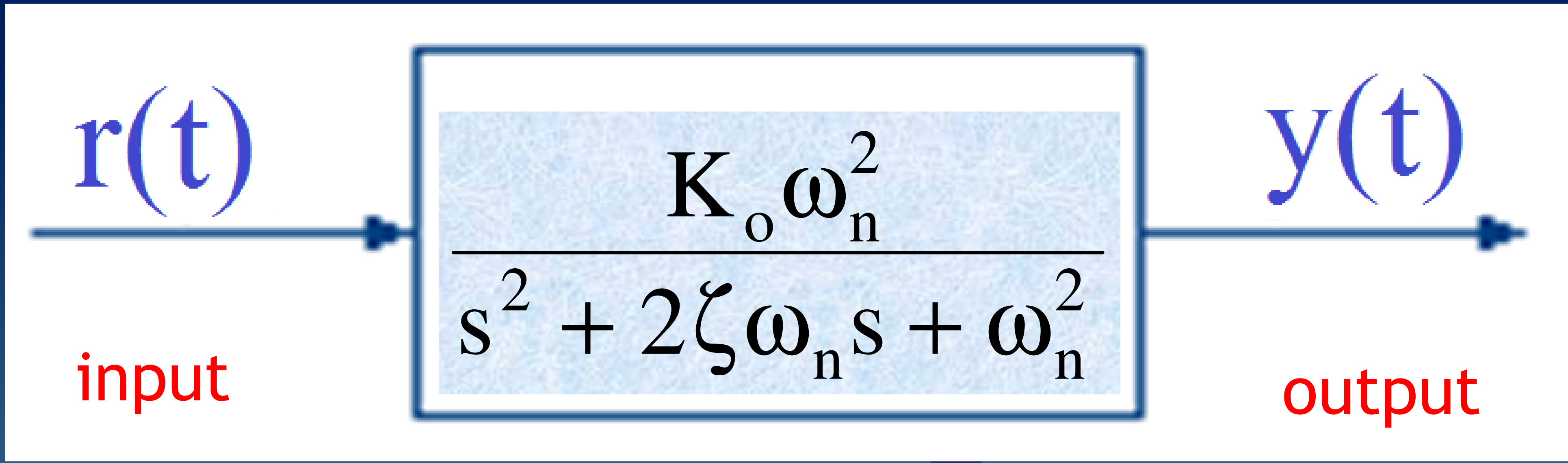
$$y(t) = K_o\left[1 - e^{-\zeta\omega_n t}\left(1+\omega_n t\right)\right]$$



Finally, let us now see $y(t)$, the output to the unit step for case (c)

c) $\zeta > 1$ *(over damping)*

Time domain analysis - 2nd order systems



$$y(t) = \mathscr{L}^{-1}[Y(s)]$$

In the case $\zeta > 1$ (*over damping*) the unit step response is:

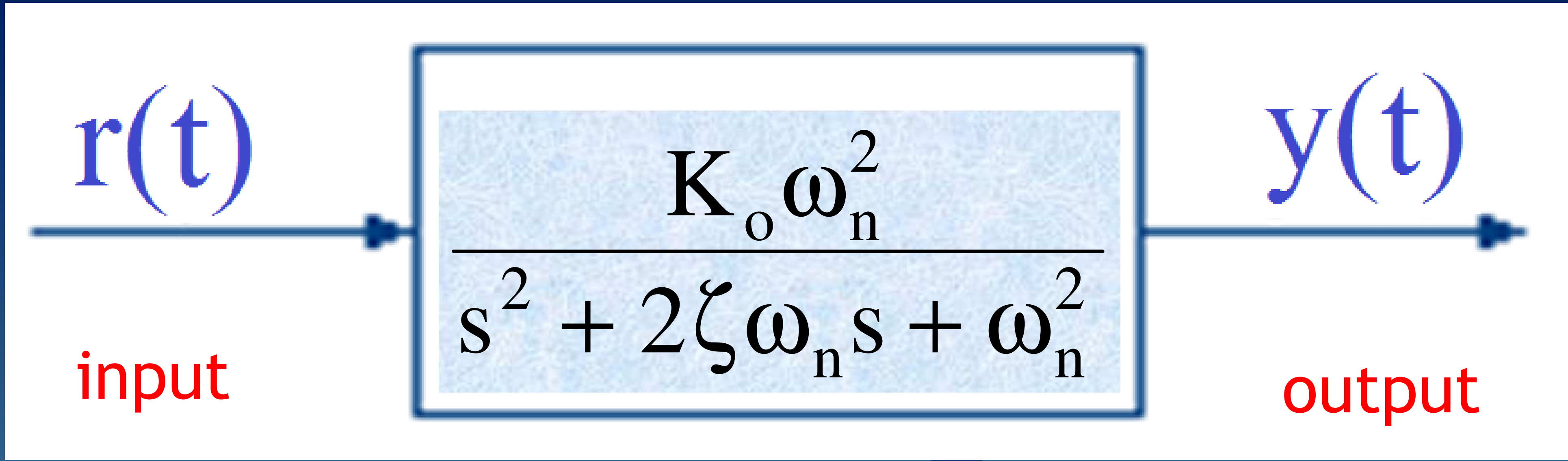
$$y(t) = K_o\left[1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\left(\frac{e^{p_1 t}}{p_1}-\frac{e^{p_2 t}}{p_2}\right)\right], \qquad t>0$$

where

$$p_{1,2} = -\zeta\omega_n \mp \omega_n\sqrt{\zeta^2-1} = -\omega_n\left(\zeta \pm \sqrt{\zeta^2-1}\right)$$

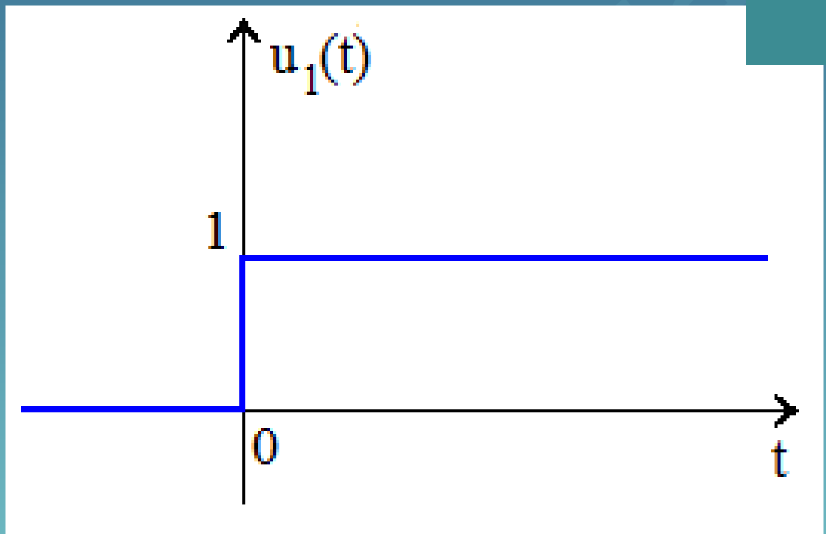
System has real poles

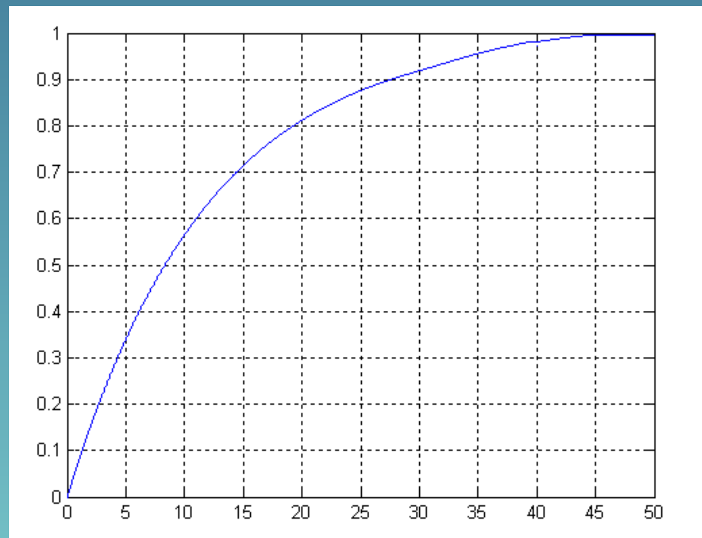
Time domain analysis - 2nd order systems



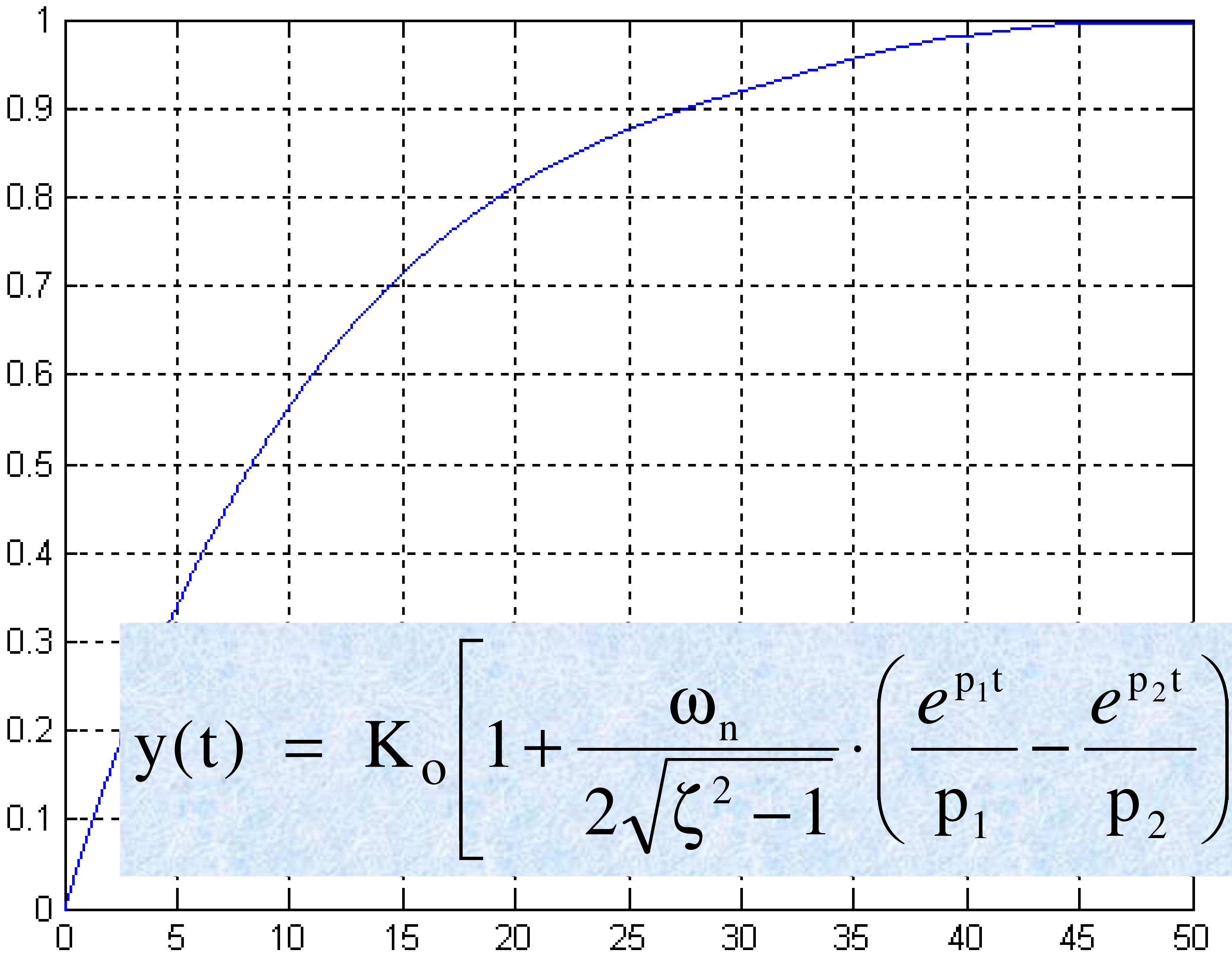
unit step response:

$$y(t) = K_o \left[1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \left(\frac{e^{p_1 t}}{p_1} - \frac{e^{p_2 t}}{p_2} \right) \right]$$



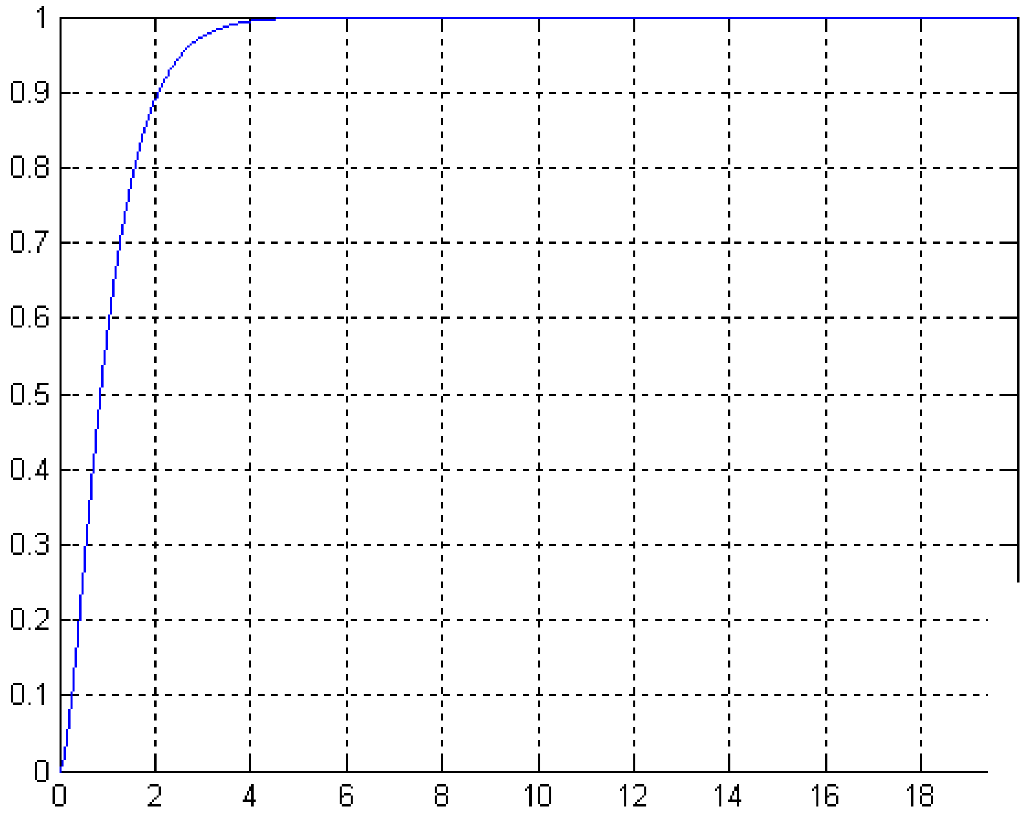


unit step response:



$$y(t) = K_o \left[1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \left(\frac{e^{p_1 t}}{p_1} - \frac{e^{p_2 t}}{p_2} \right) \right]$$

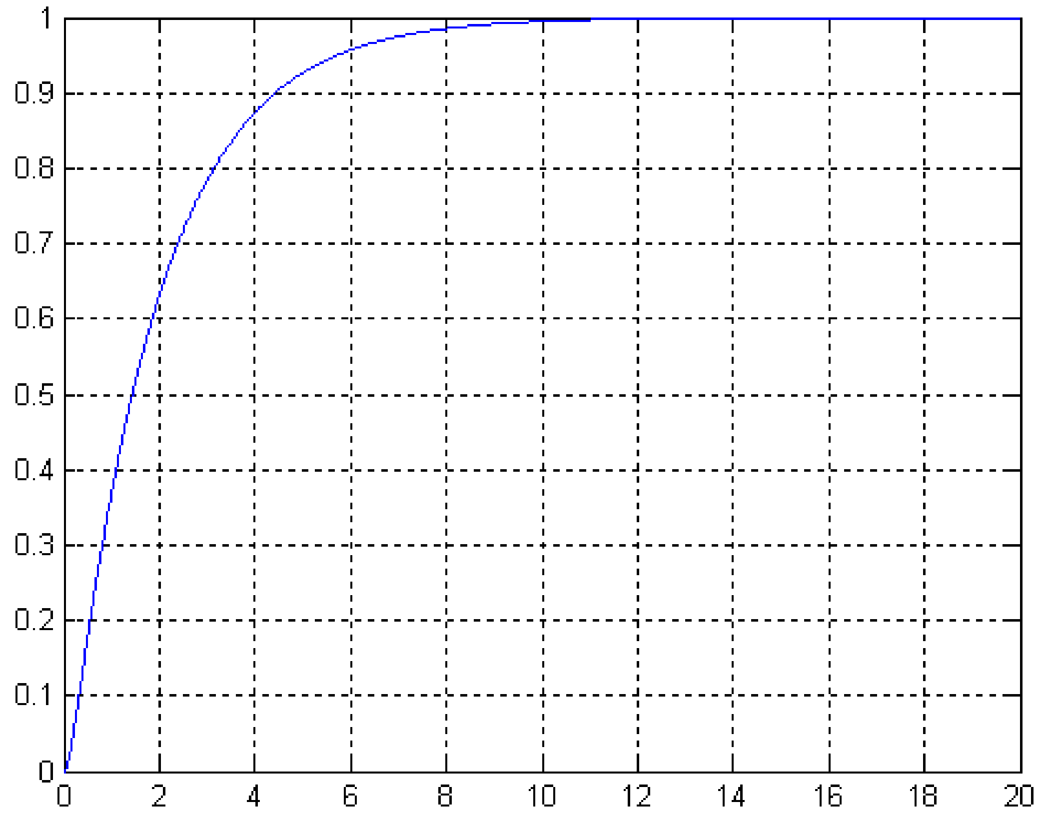
Time domain analysis - 2nd order systems



unit step
response

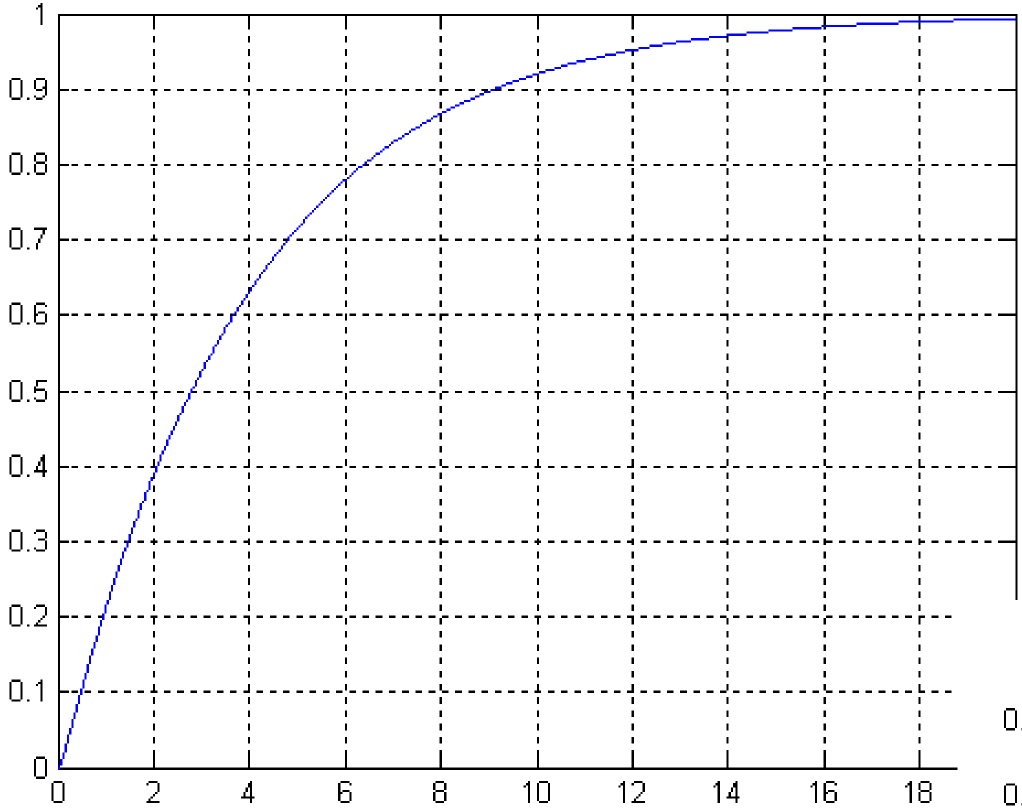
$\zeta = 1$

$\zeta = 2$



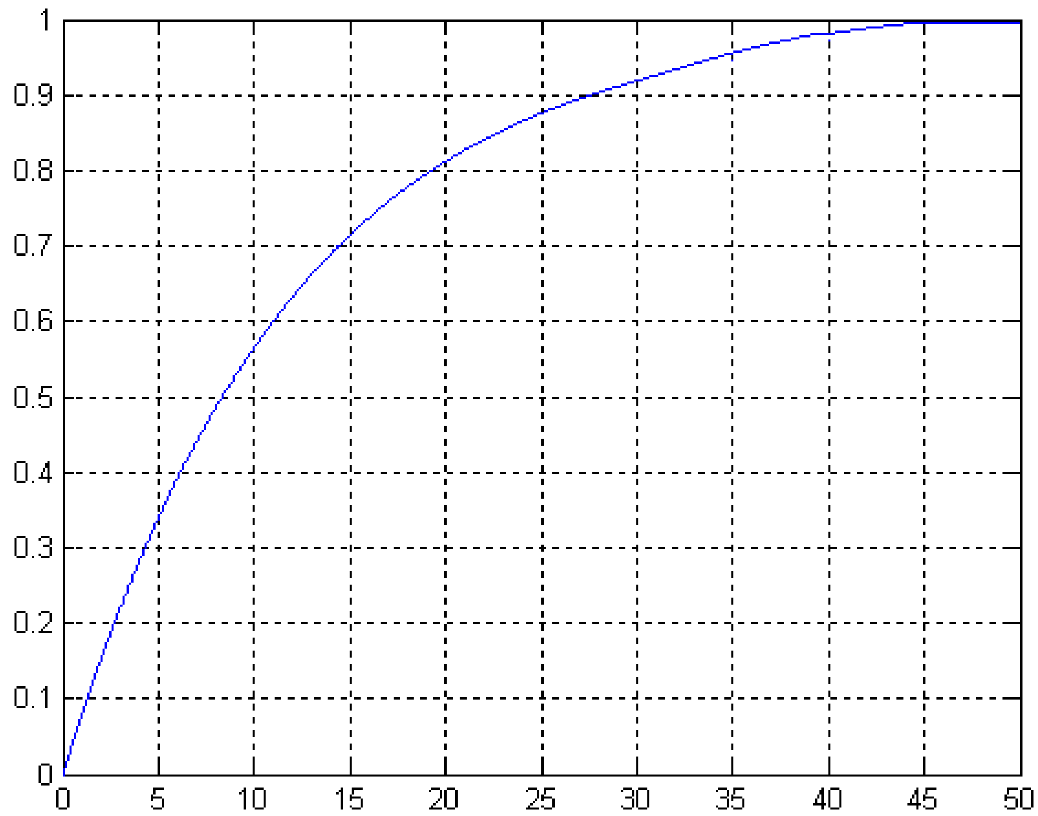
Time domain analysis - 2nd order systems

unit step
response



$\zeta = 4$

$\zeta = 12$



Let us now analyse some parameters associated with the under damping case

$$0 < \zeta < 1 \quad \text{(under damping)}$$

Time domain analysis - 2nd order systems

In the case $0 < \zeta < 1$,
the unit step response

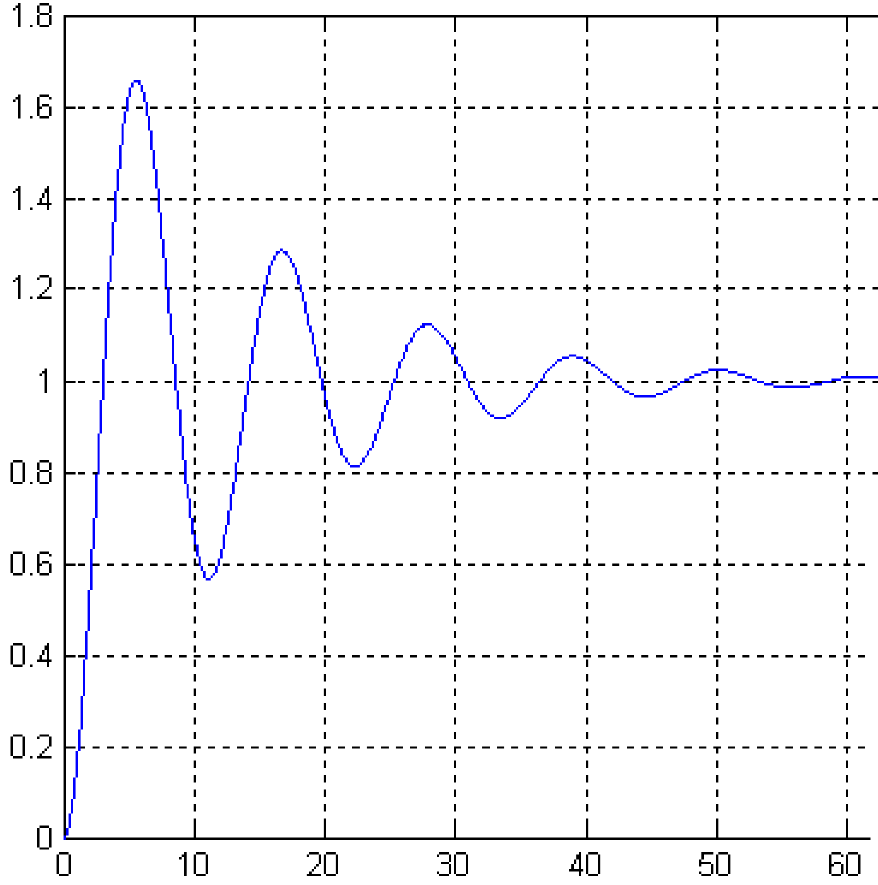
$$y(t) = K_o \left[1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \operatorname{sen} \omega_d t \right) \right], \quad t > 0$$

can have many different forms, depending on the values of ζ (*damping coefficient*), ω_n (*natural frequency*) e K_o (*gain*)

Observe that ω_d depends on ζ and ω_n

$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$$ (*damping frequency*)

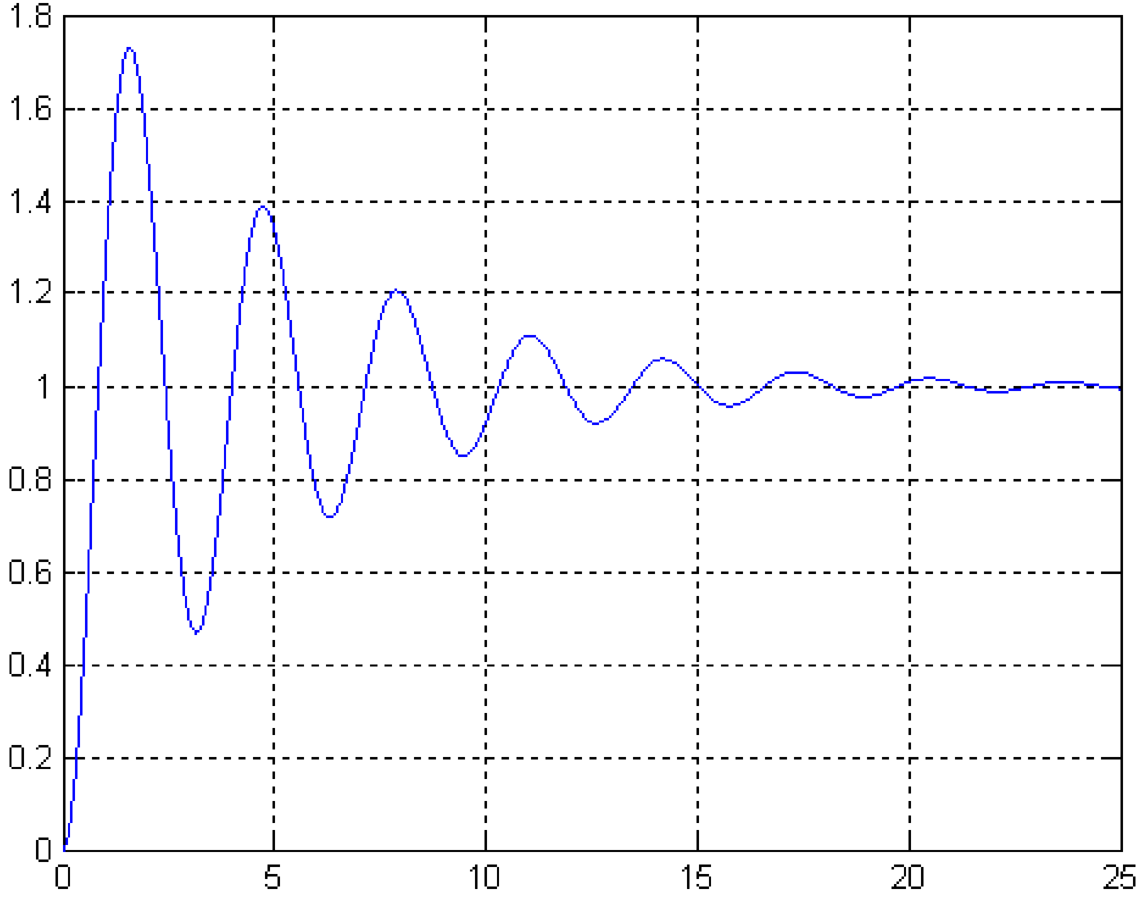
Time domain analysis - 2nd order systems



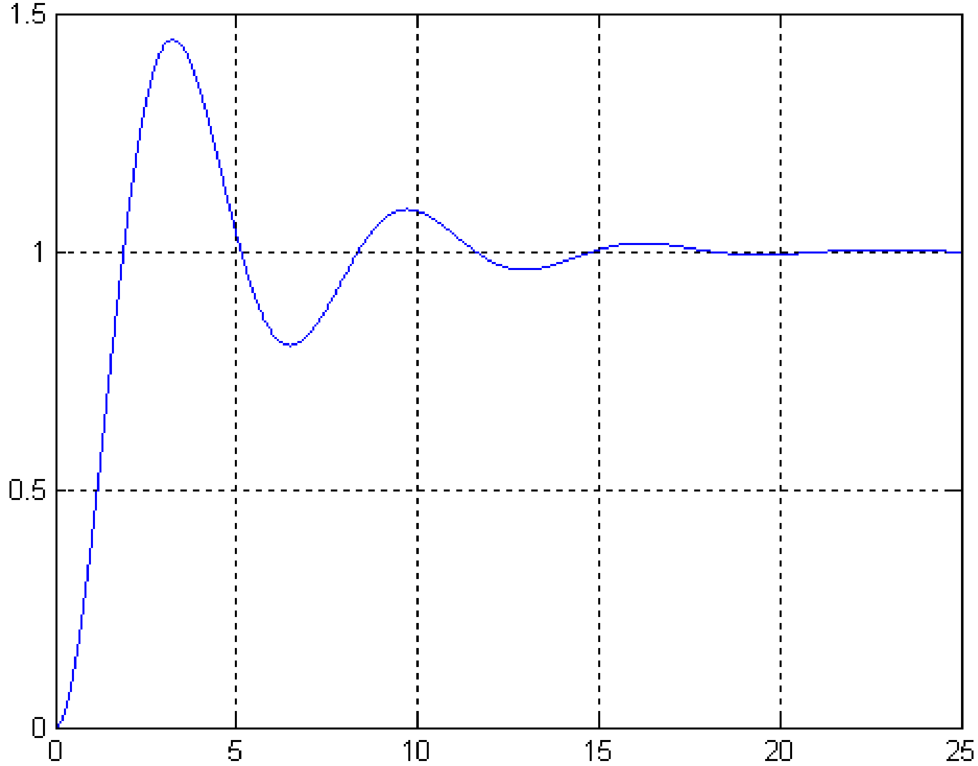
unit step
response

ζ = 0,132
ω_n = 0,57

ζ = 0,1
ω_n = 2



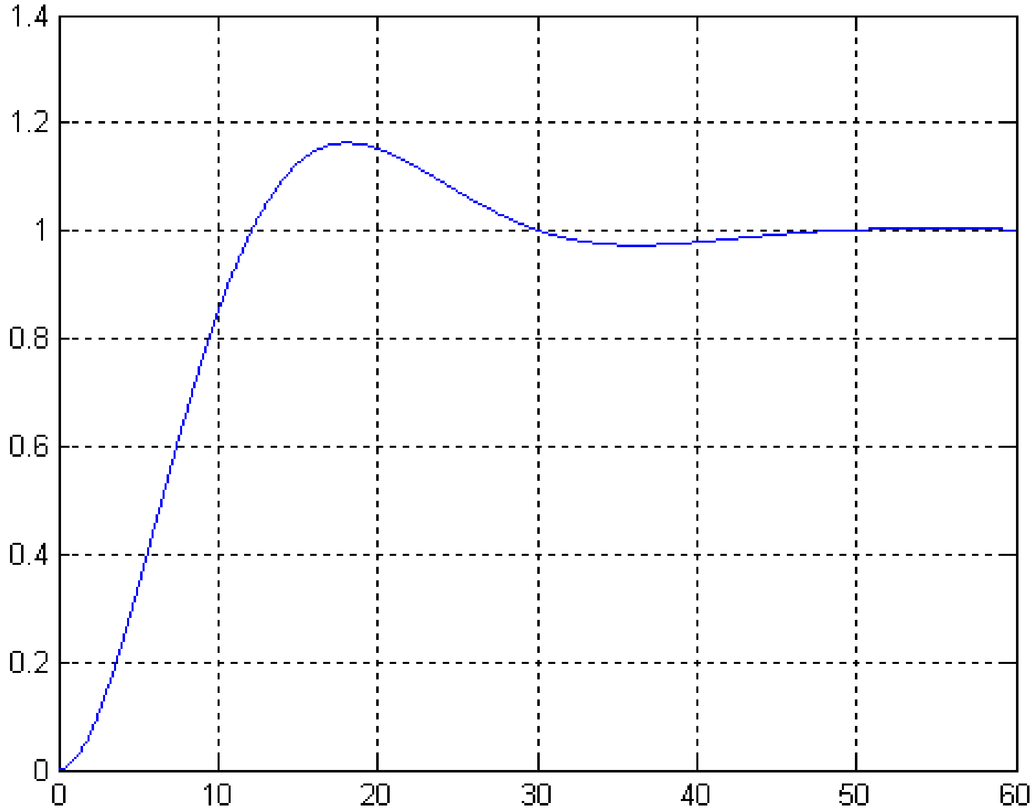
Time domain analysis - 2nd order systems



unit step
response

$\zeta = 0{,}25$

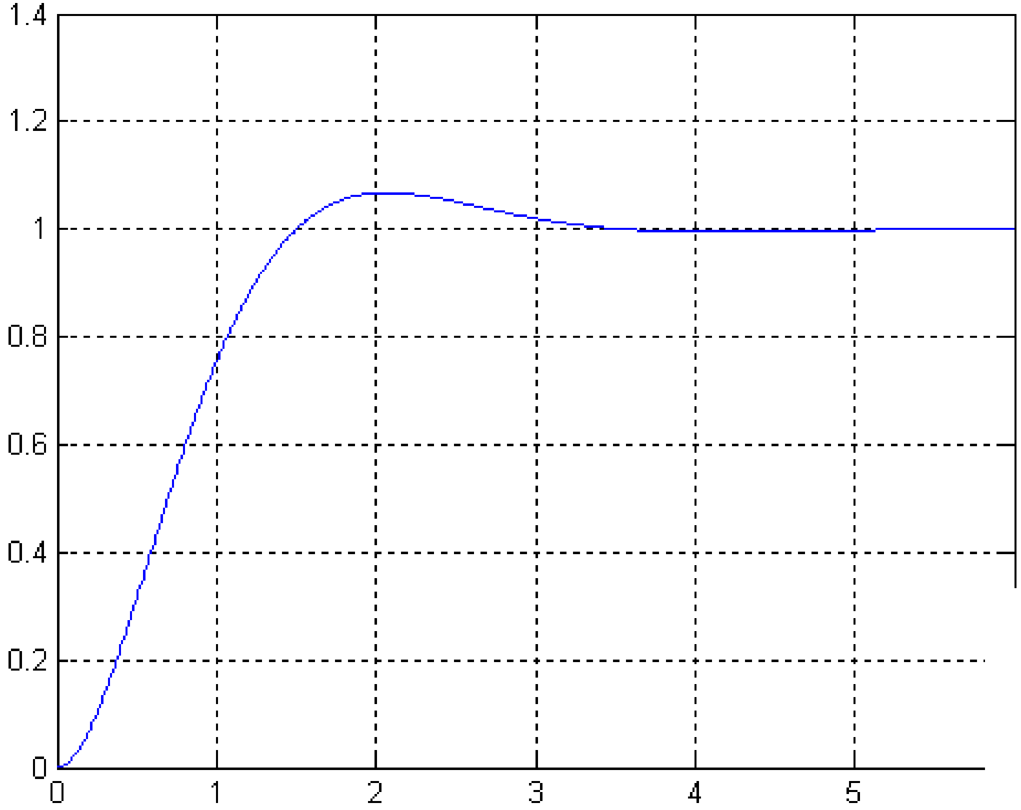
$\omega_n = 1$



$\zeta = 0{,}5$

$\omega_n = 0{,}2$

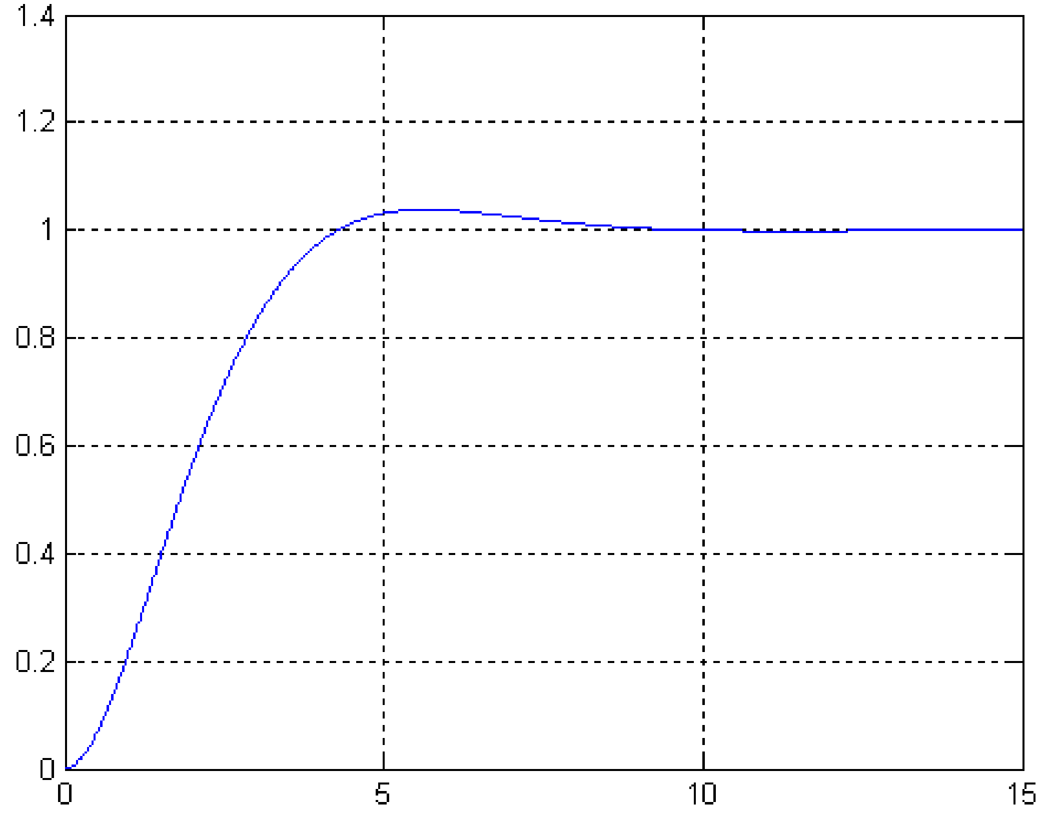
Time domain analysis - 2nd order systems



unit step response

$\zeta = 0{,}65$

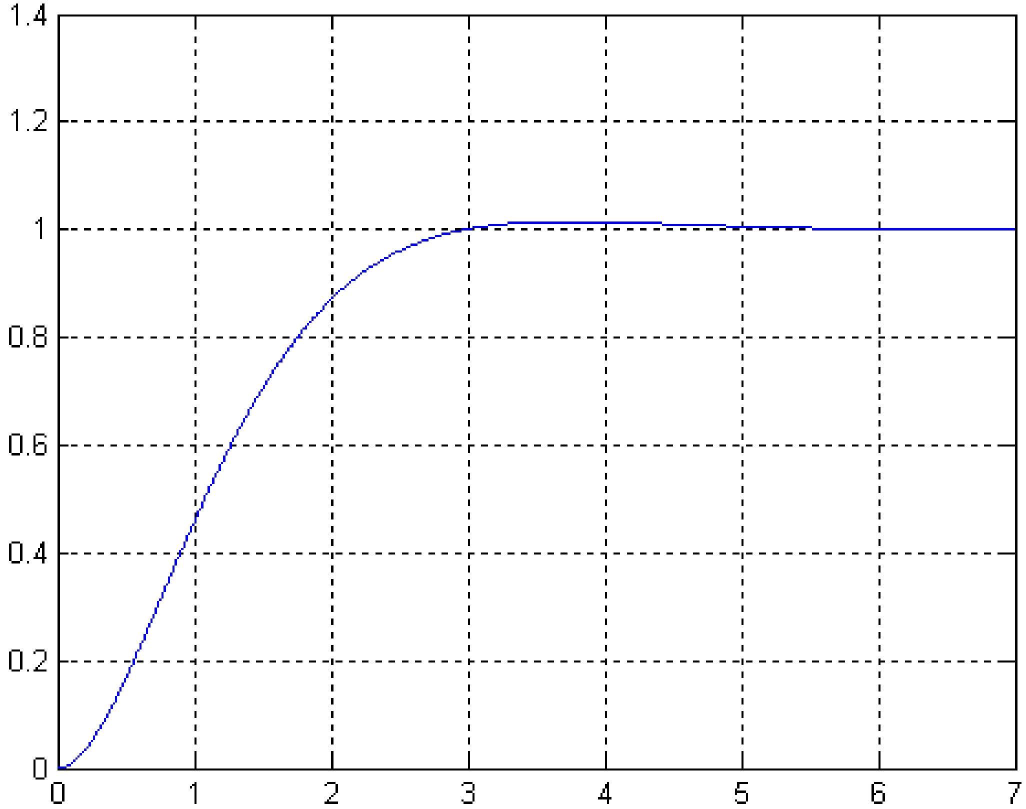
$\omega_n = 2$



$\zeta = 0{,}72$

$\omega_n = 0{,}8$

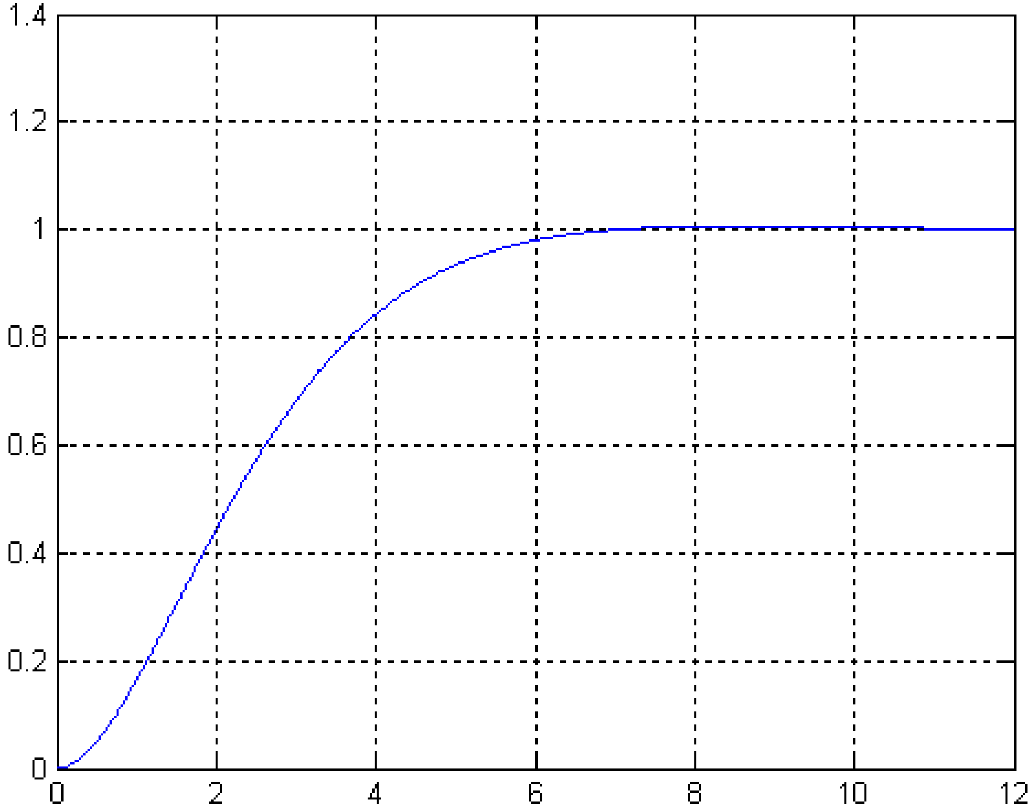
Time domain analysis - 2nd order systems



unit step
response

$\zeta = 0,8$
$\omega_n = 1,4$

$\zeta = 0,85$
$\omega_n = 0,7$

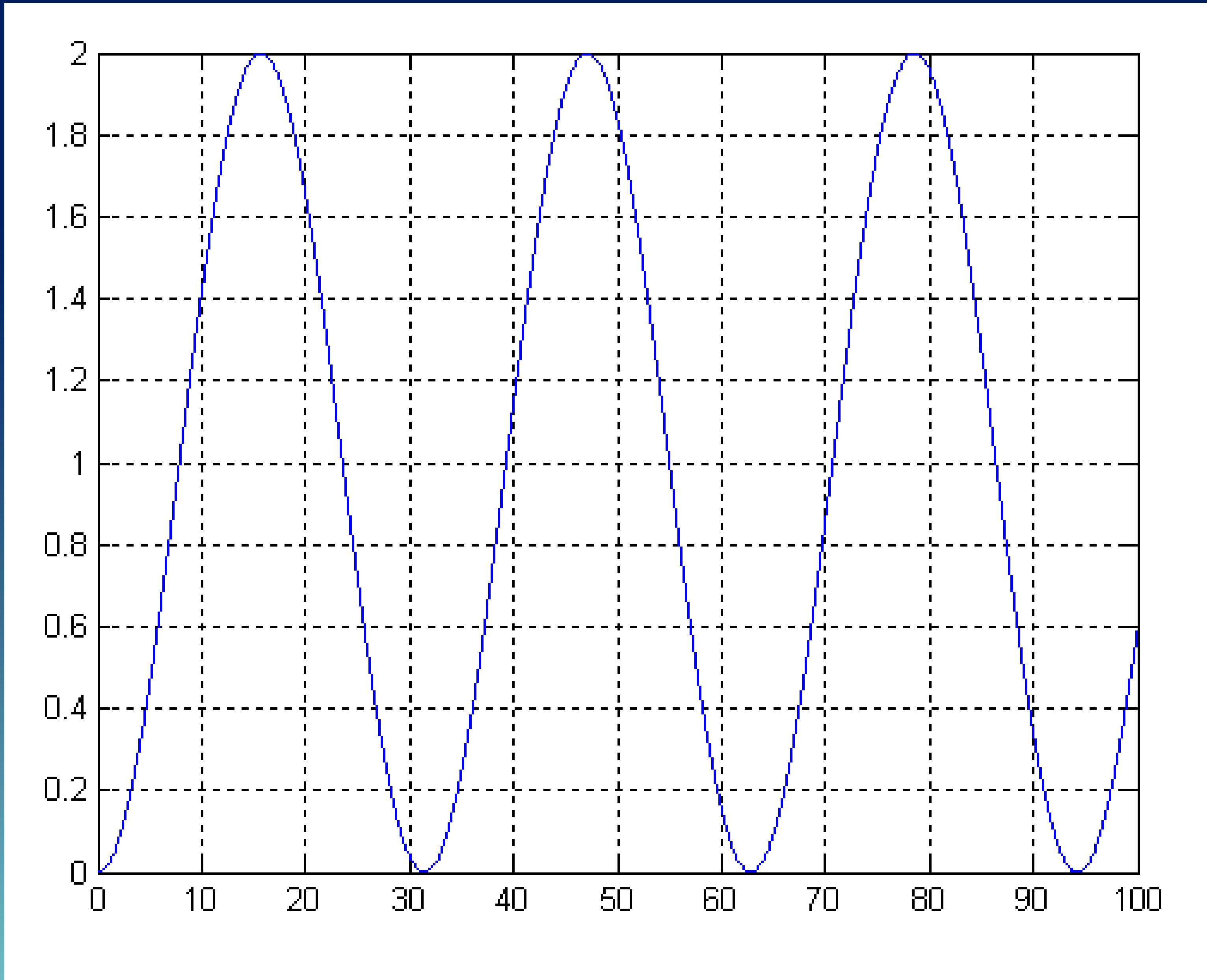


Time domain analysis - 2nd order systems

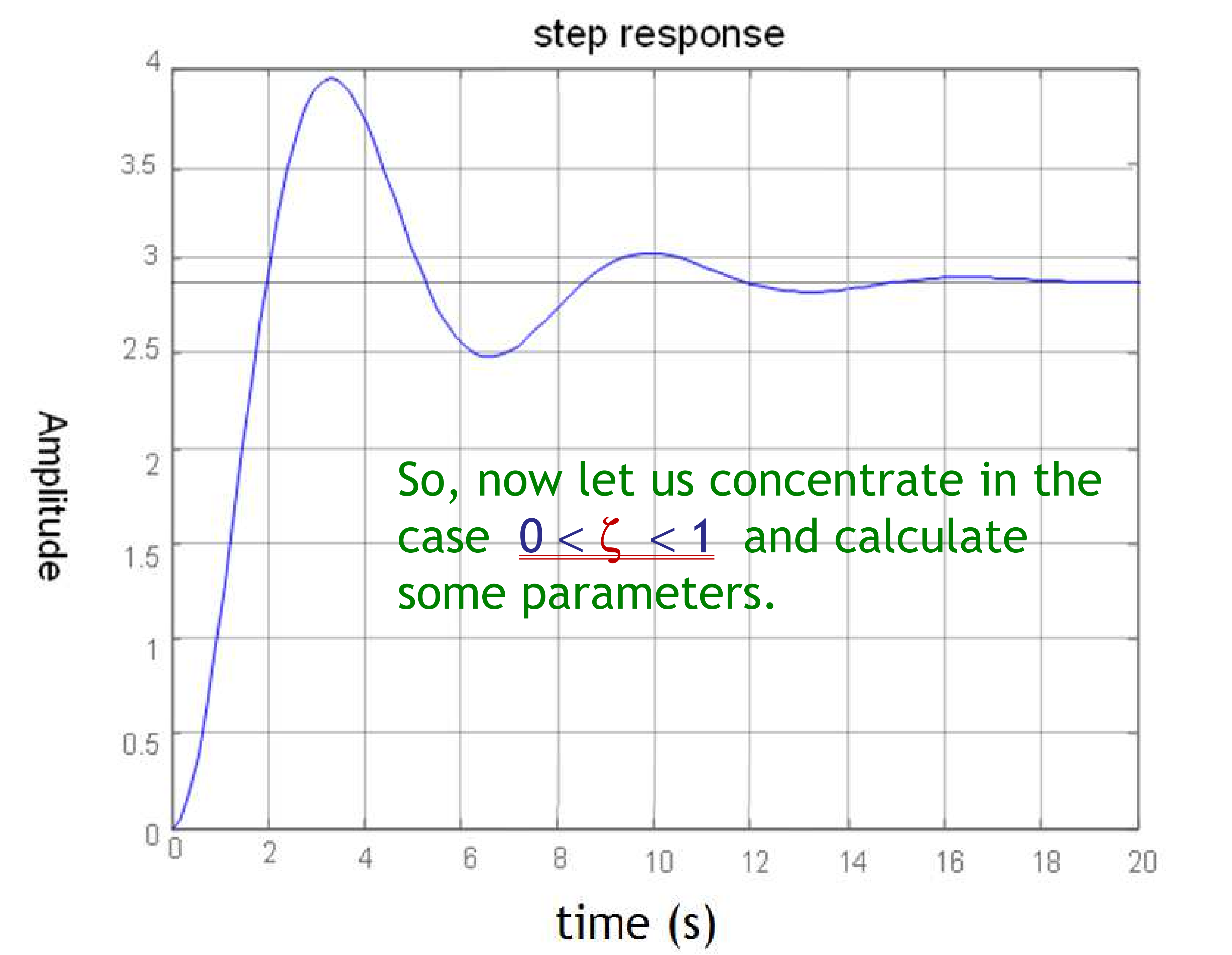
unit step
response

$\zeta = 0$

$\omega_n = 0{,}2$

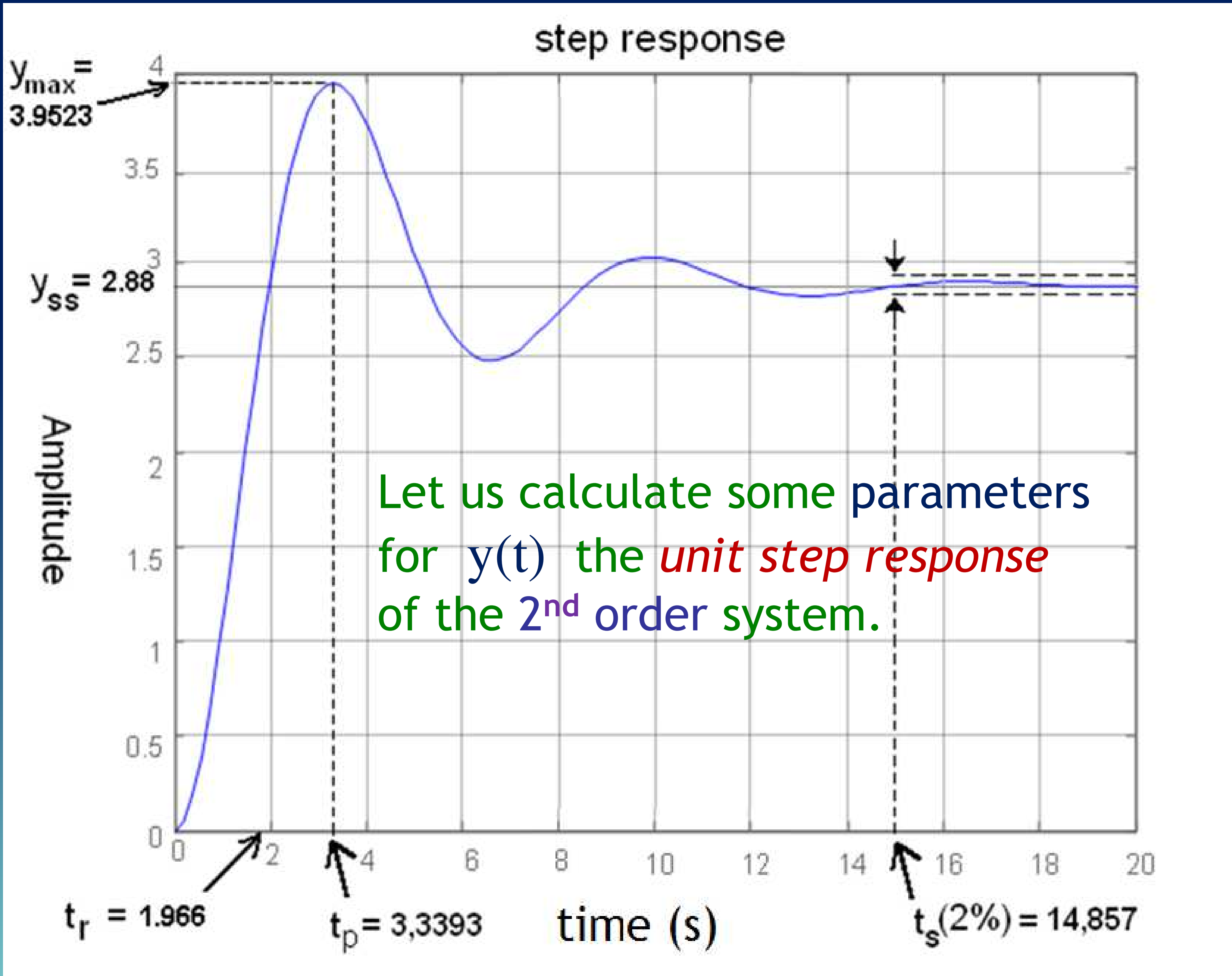


Time domain analysis - 2^{nd} order systems



So, now let us concentrate in the case $0 < \zeta < 1$ and calculate some parameters.

Time domain analysis - 2nd order systems



Let us calculate some parameters for $y(t)$ the *unit step response* of the 2nd order system.

steady state output

y_{ss}

y_{ss} = steady state response

(*steady state output*)

$$Y(s) = \frac{K_o\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot R(s)$$

$$R(s) = \frac{1}{s}$$

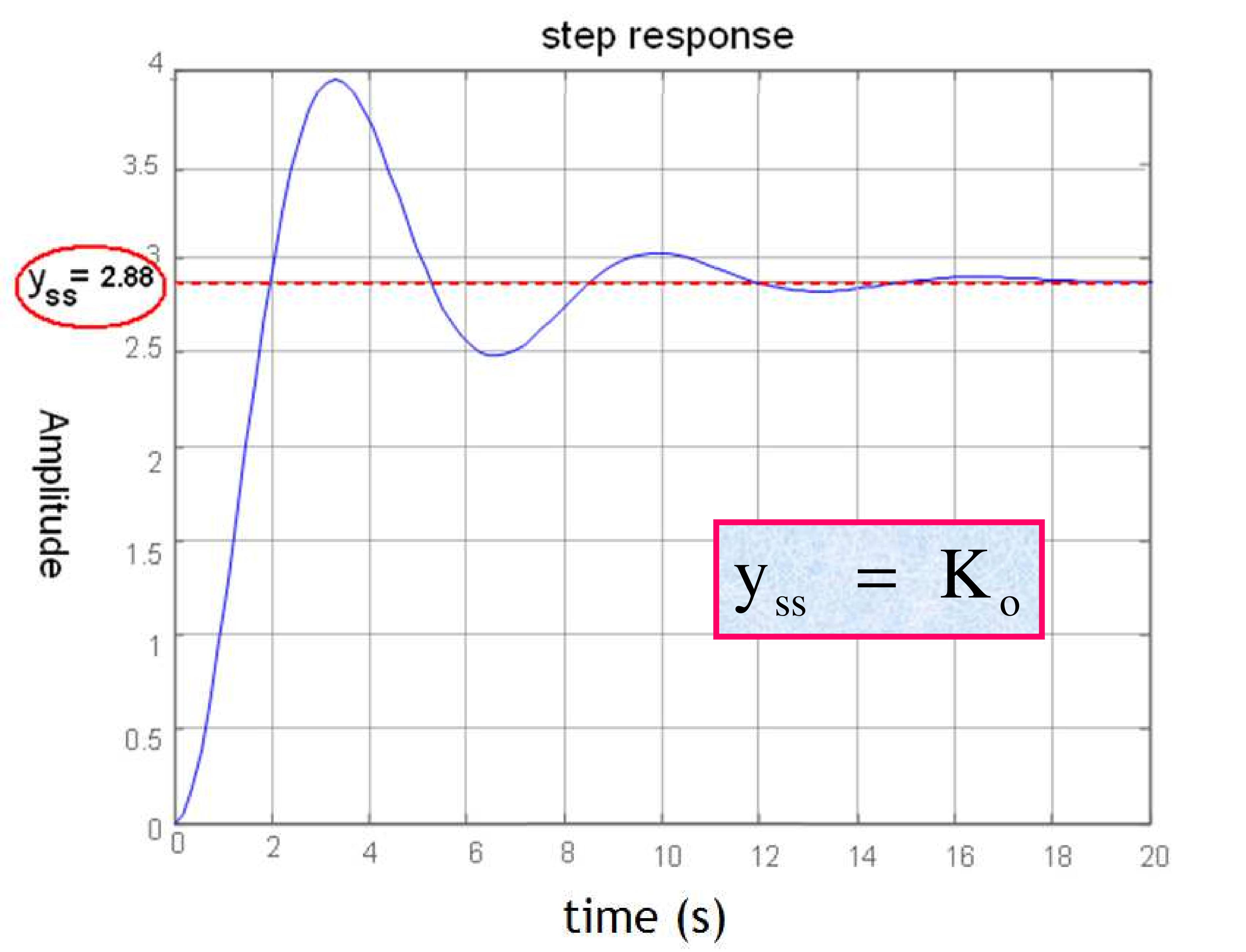
$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to o} s \cdot Y(s) =$$

$$= \lim_{s\to 0} \frac{K_o\,\omega_n^2\, s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} =$$

$$= K_o$$

$$y_{ss} = K_o$$

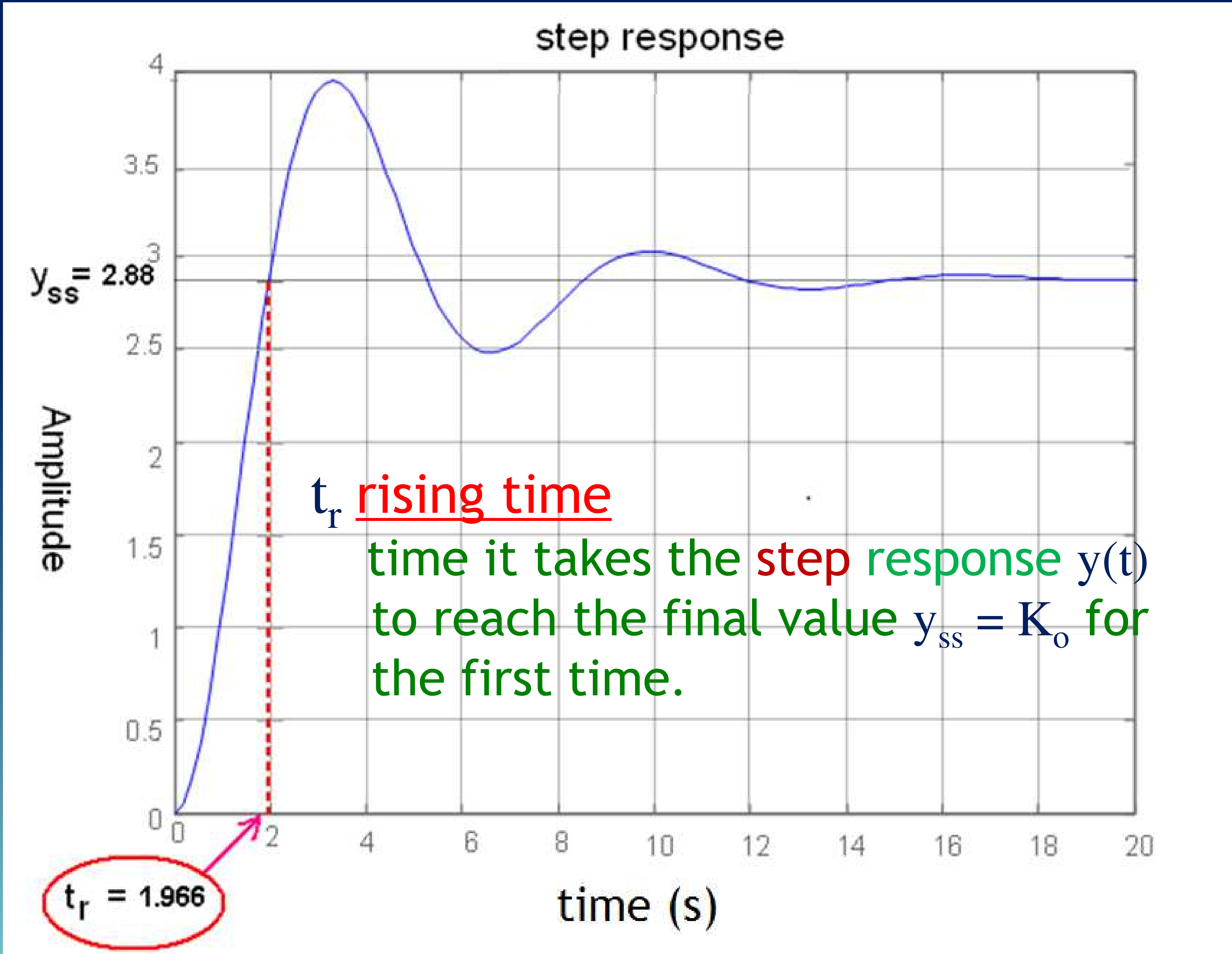
Time domain analysis - 2nd order systems



rising time

t_r

Time domain analysis - 2nd order systems



t_r rising time

time it takes the step response $y(t)$ to reach the final value $y_{ss} = K_o$ for the first time.

t_r = rising time

Is the instant of time that $y(t)$ reaches the final value K_o for the first time.

$$y(t_r) = K_o \left[1 - e^{-\zeta\omega_n t_r} \left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \operatorname{sen}\omega_d t_r \right) \right] = K_o$$

$$e^{-\zeta\omega_n t_r} \left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \operatorname{sen}\omega_d t_r \right) = 0$$

$$\operatorname{tg}(\omega_d t_r) = \frac{-\sqrt{1-\zeta^2}}{\zeta} = \frac{-\omega_d}{\zeta\omega_n}$$

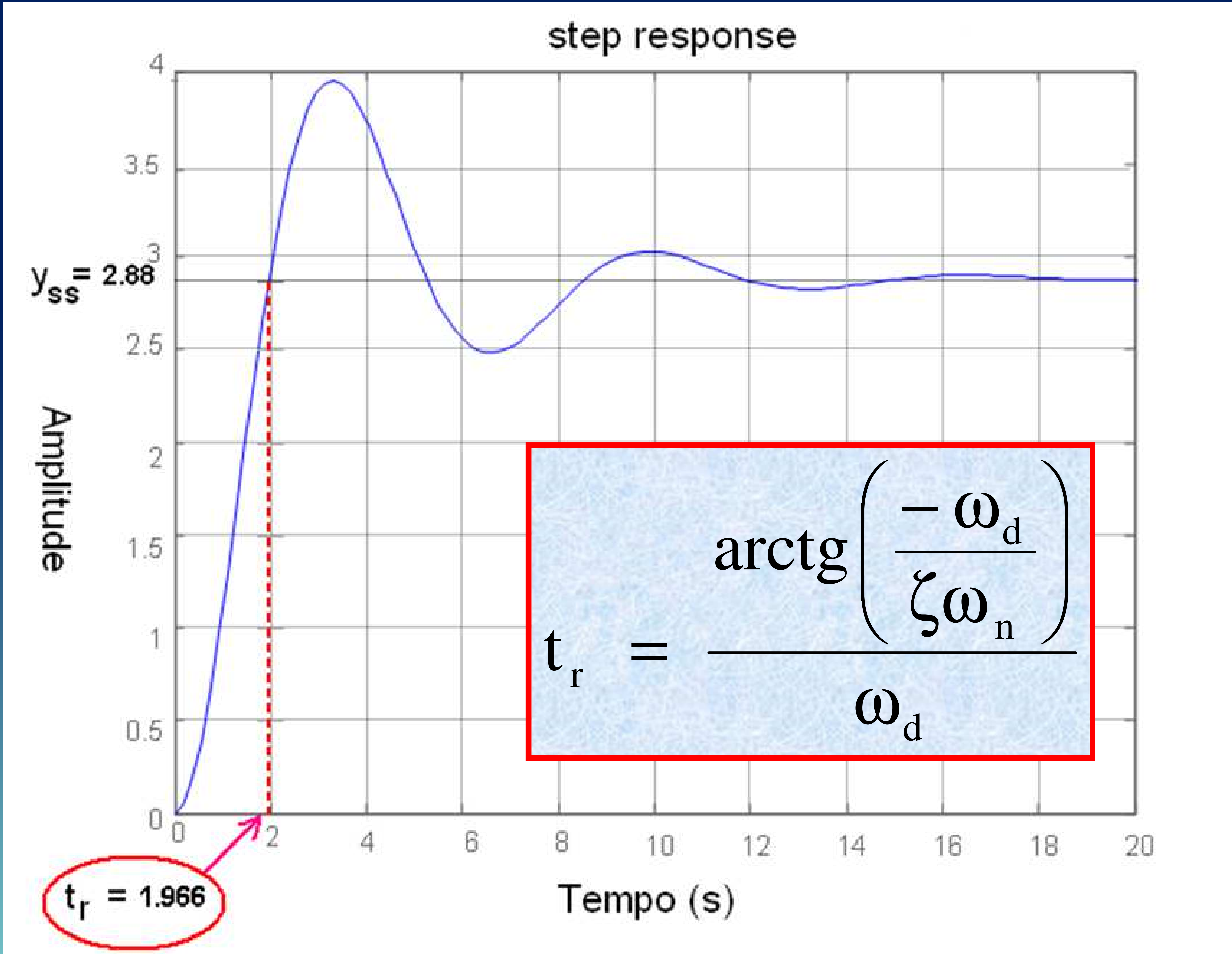
t_r = rising time

Depends on the values of ζ *(damping coefficient)*, and of ω_n *(natural frequency)*

$$t_r = \frac{\operatorname{arctg}\left(\dfrac{-\omega_d}{\zeta\omega_n}\right)}{\omega_d} = \frac{\operatorname{arctg}\left(\dfrac{-\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_d}$$

$$t_r = \operatorname{arctg}(-\omega_d/\zeta\omega_n) / \omega_d$$

Time domain analysis - 2nd order systems

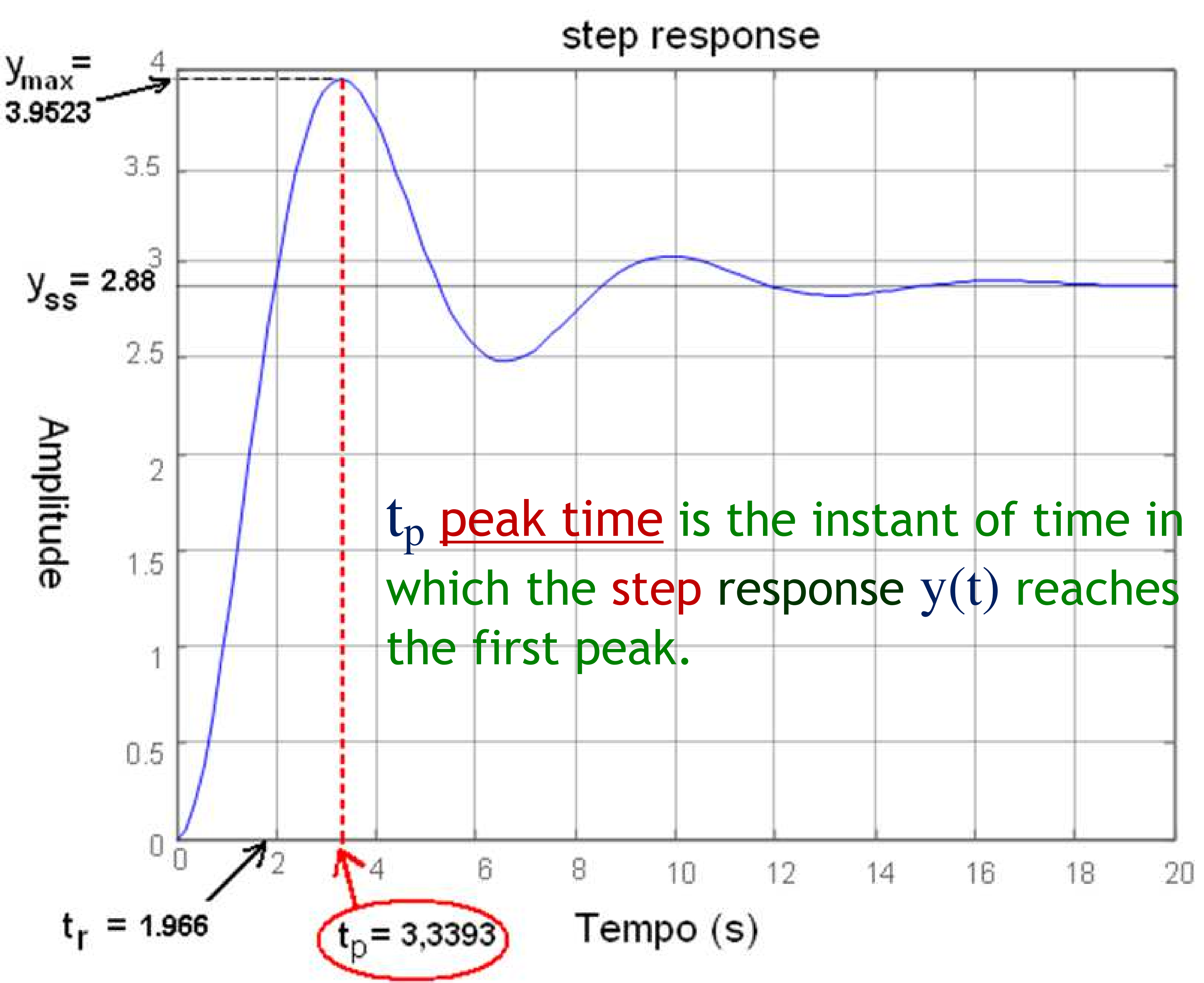


$$t_r = \frac{\operatorname{arctg}\left(\frac{-\omega_d}{\zeta\omega_n}\right)}{\omega_d}$$

peak time

t_p

Time domain analysis - 2nd order systems



t_p peak time is the instant of time in which the step response $y(t)$ reaches the first peak.

peak time

$y_{ss} = K_o$ *(gain)*

ζ *(damping coefficient),*

ω_n *(natural frequency)*

$$y' = \frac{dy}{dt} = K_o\left[\zeta\omega_n \cdot e^{-\zeta\omega_n t}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot \text{sen}\,\omega_d t_r\right) + \right.$$

$$\left. - e^{-\zeta\omega_n t}\left(-\omega_d \text{sen}\,\omega_d t_r + \zeta\frac{\omega_d}{\sqrt{1-\zeta^2}}\cdot\cos\omega_d t_r\right)\right]$$

ω_n

peak time

$$\frac{dy}{dt} = K_o \cdot e^{-\zeta\omega_n t}\left(\zeta\omega_n \cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} \cdot \operatorname{sen}\omega_d t \; + \right.$$

$$\left. + \, \omega_d \operatorname{sen}\omega_d t - \zeta\omega_n \cdot \cos\omega_d t \right)$$

$$= K_o \cdot e^{-\zeta\omega_n t} \cdot \operatorname{sen}\omega_d t \cdot \left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \frac{\omega_n (1-\zeta^2)}{\sqrt{1-\zeta^2}} \right)$$

$$= K_o \cdot e^{-\zeta\omega_n t} \cdot \operatorname{sen}\omega_d t \cdot \frac{\omega_n}{\sqrt{1-\zeta^2}} = 0$$

peak time

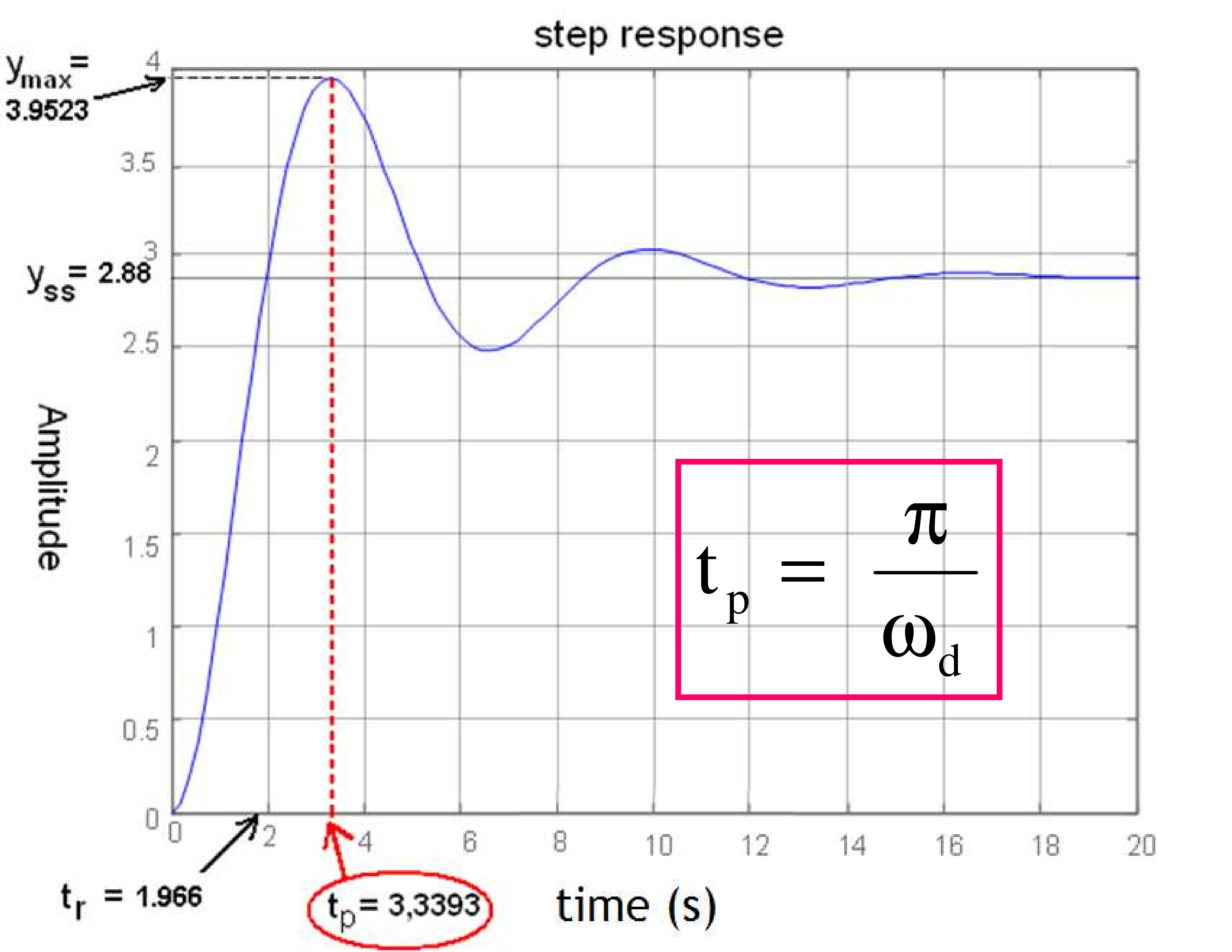
$$\frac{dy}{dt} = 0 \quad \Rightarrow \quad \text{sen}\, \omega_d t = 0$$

$$\Rightarrow \quad \omega_d t = 0, \quad \pi, \quad 2\pi, \quad 3\pi, \quad \cdots$$

$$\Rightarrow \quad t_p = \frac{\pi}{\omega_d}$$

$$t_p = \pi / \omega_d$$

Time domain analysis - 2nd order systems

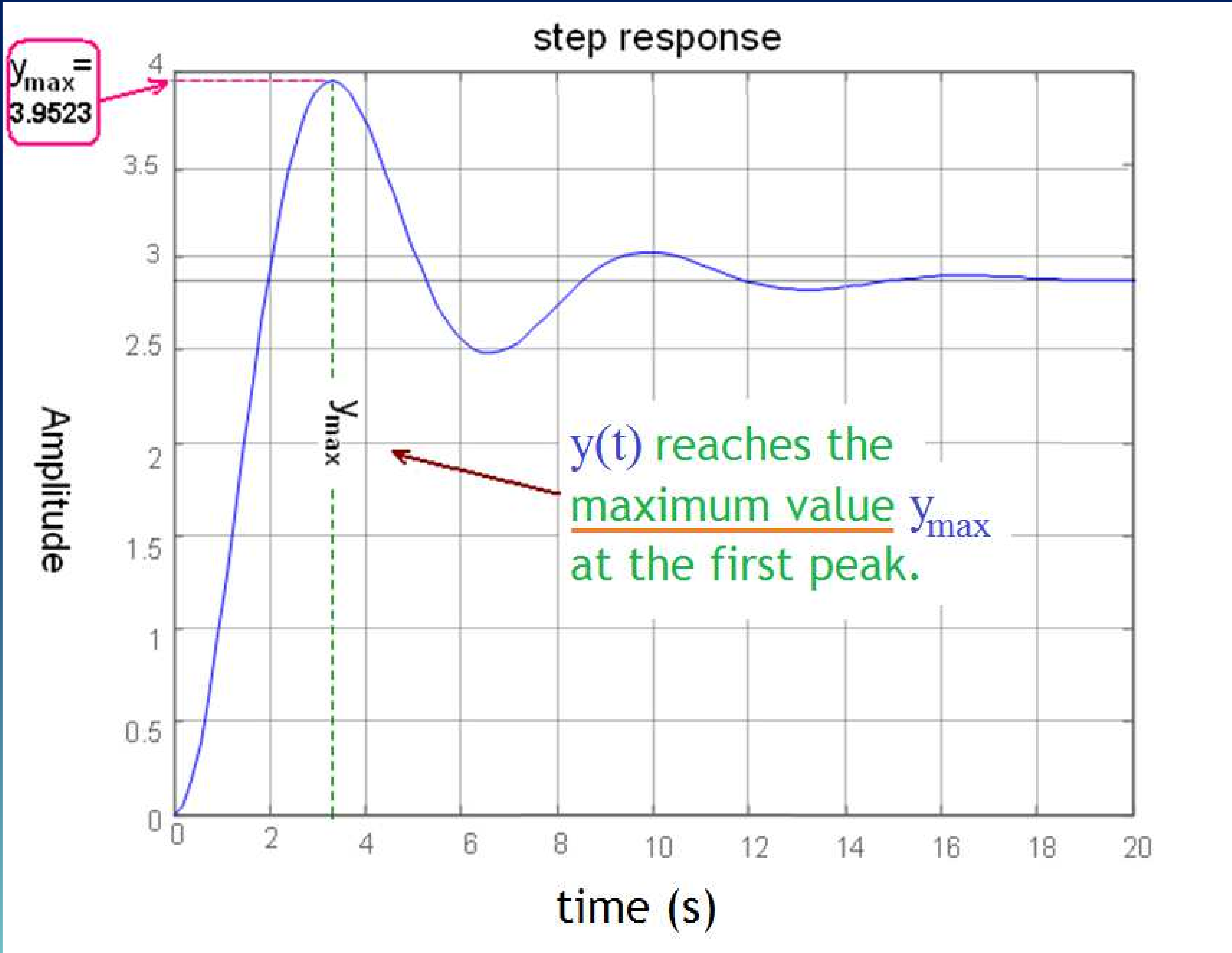


$$t_p = \frac{\pi}{\omega_d}$$

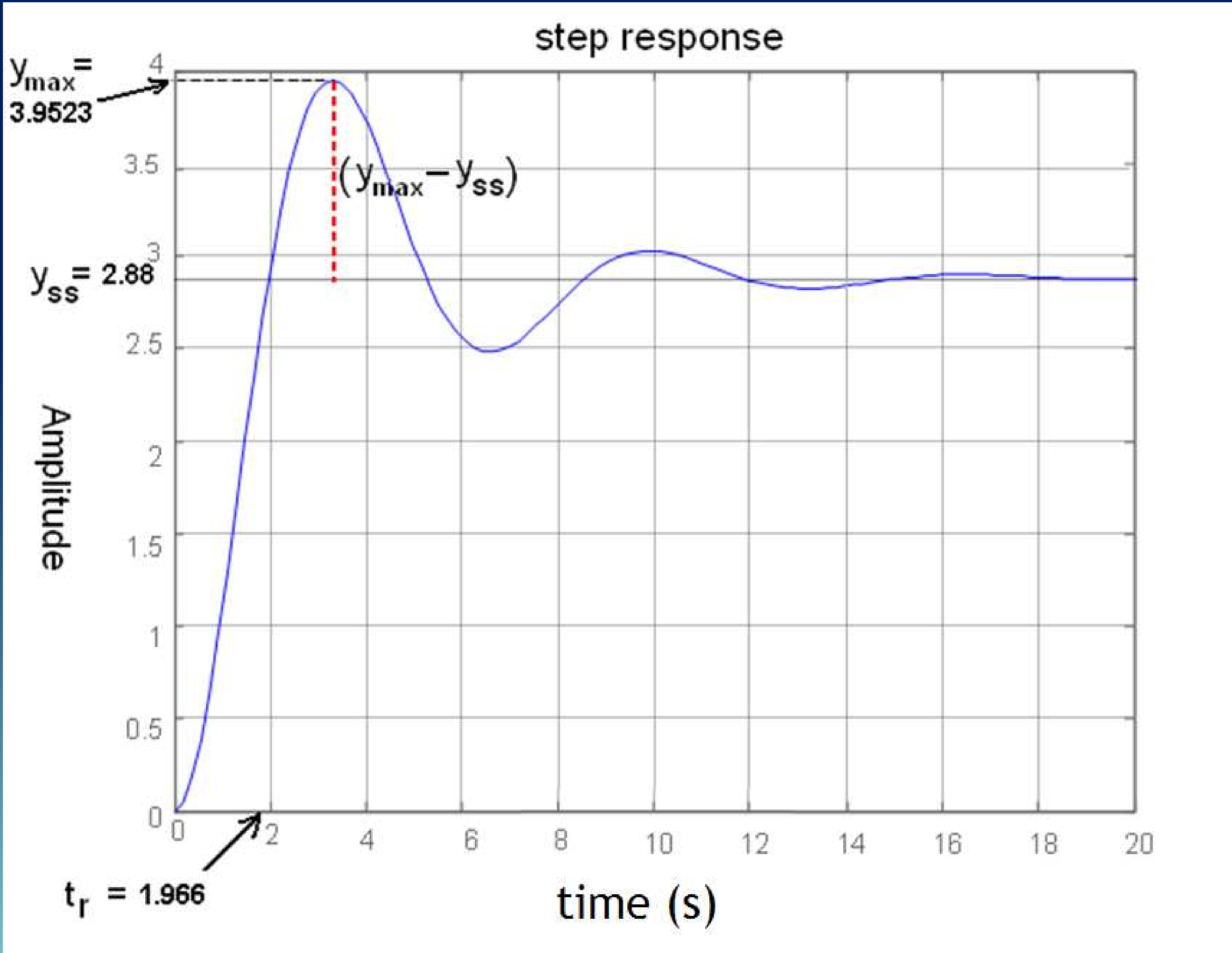
overshoot

M_p

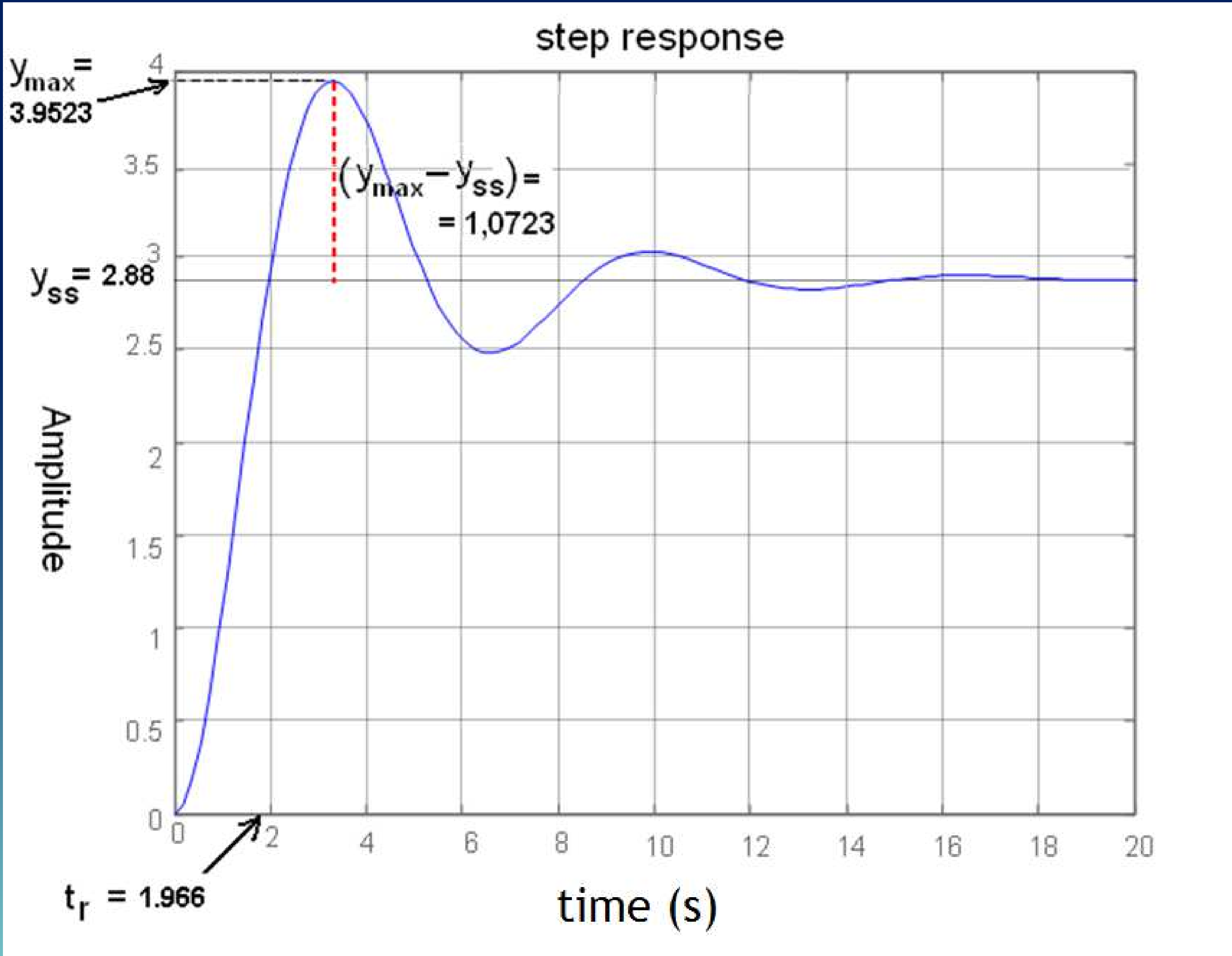
Time domain analysis – 2nd order systems



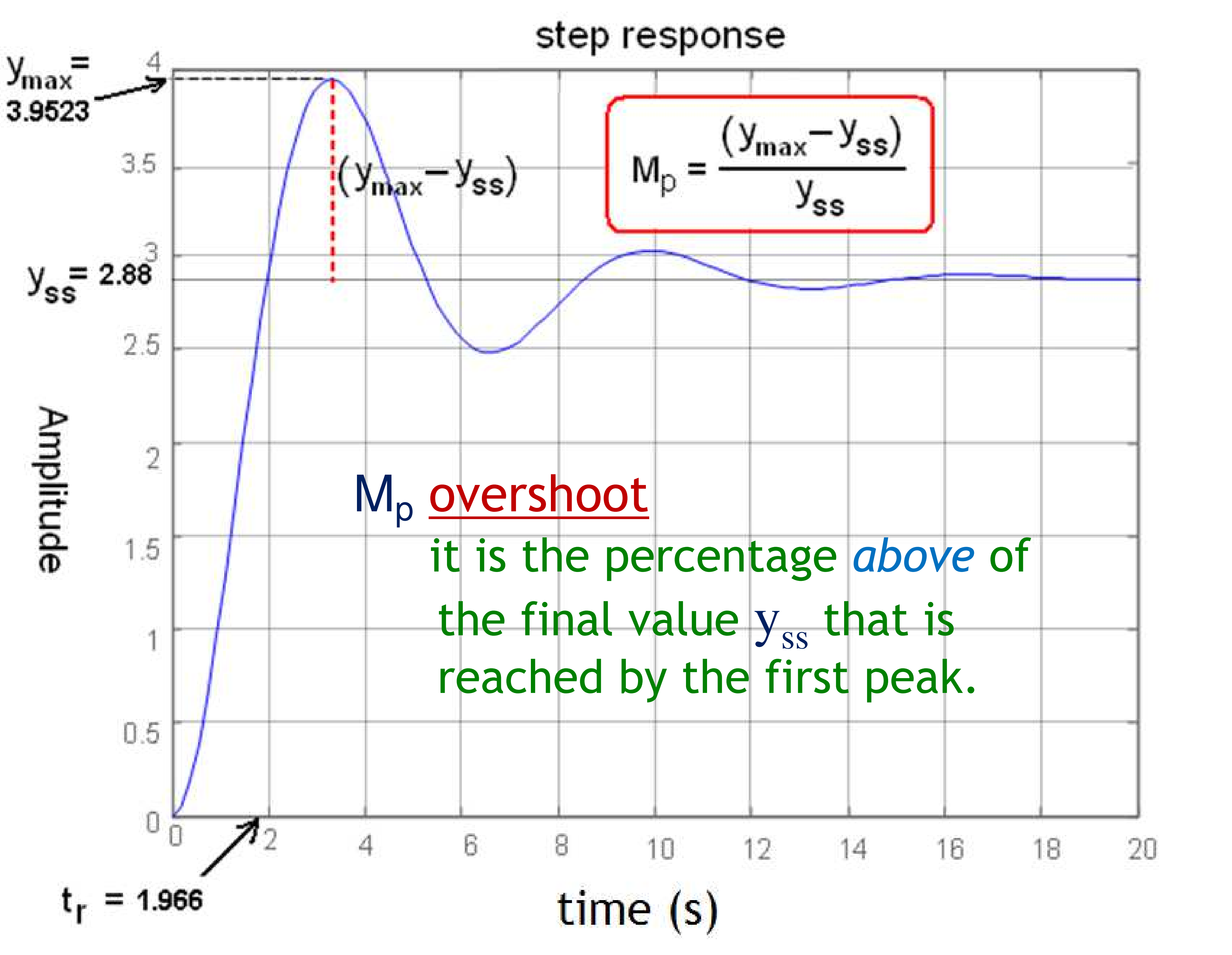
Time domain analysis – 2nd order systems



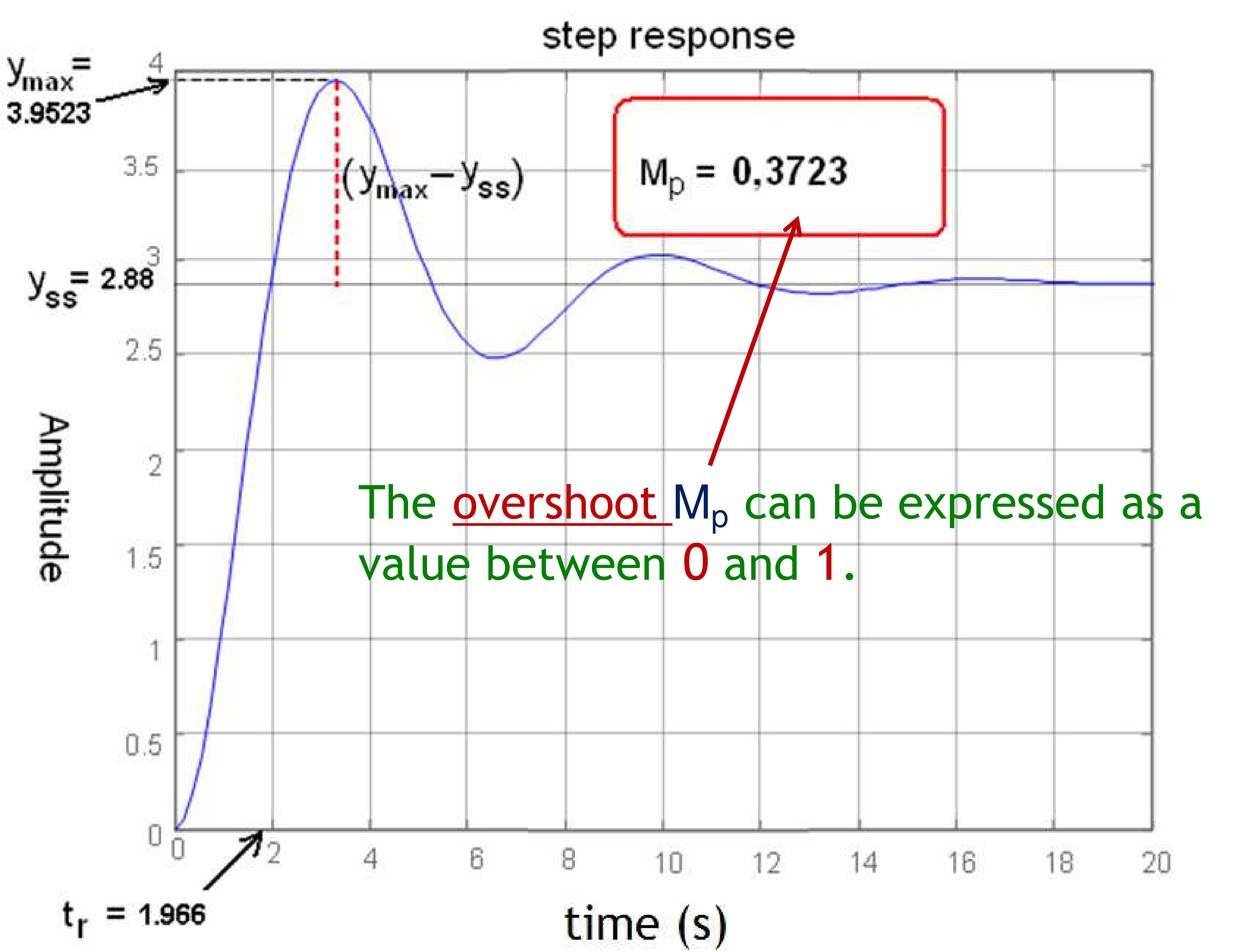
Time domain analysis - 2nd order systems



Time domain analysis - 2nd order systems

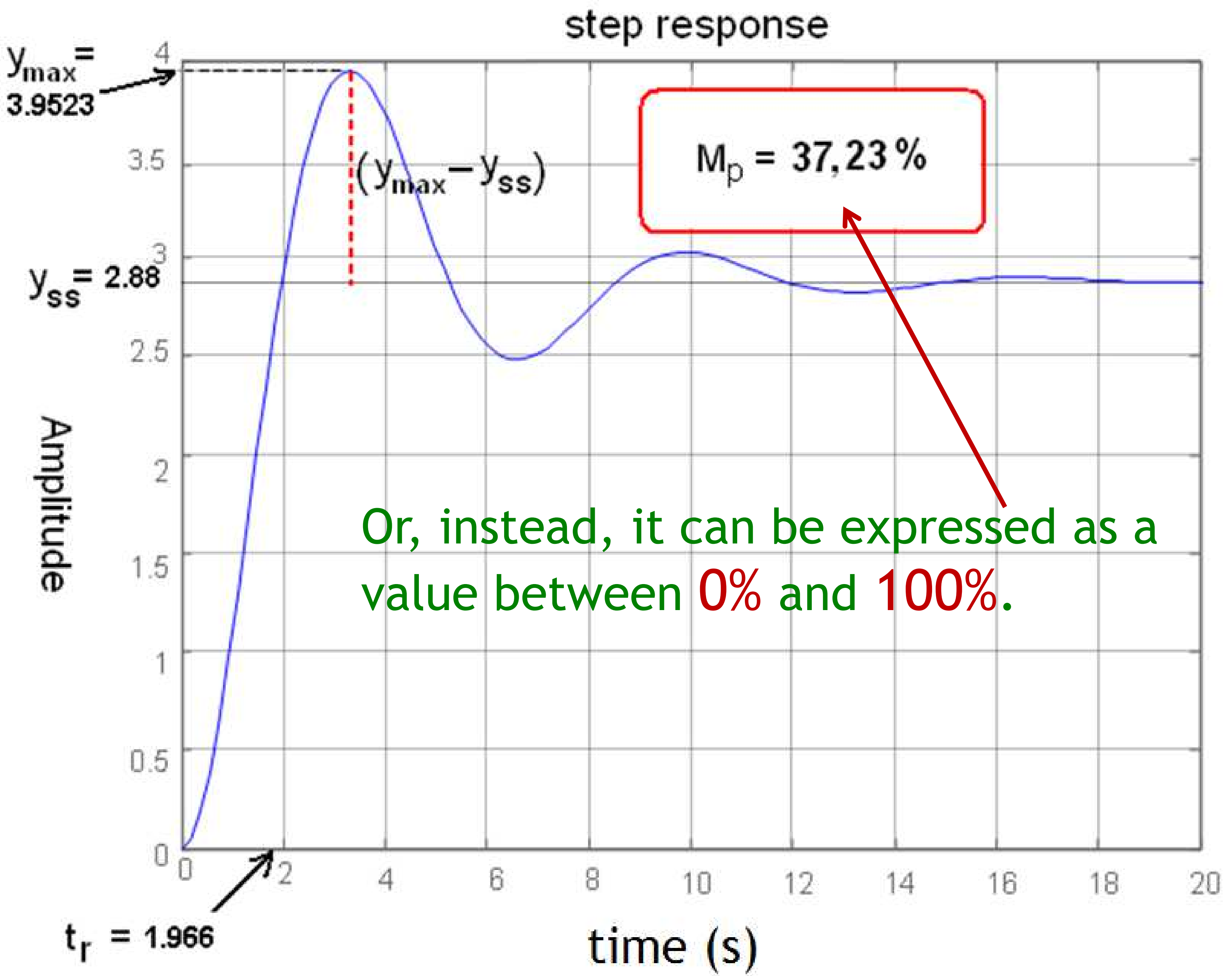


Time domain analysis - 2nd order systems



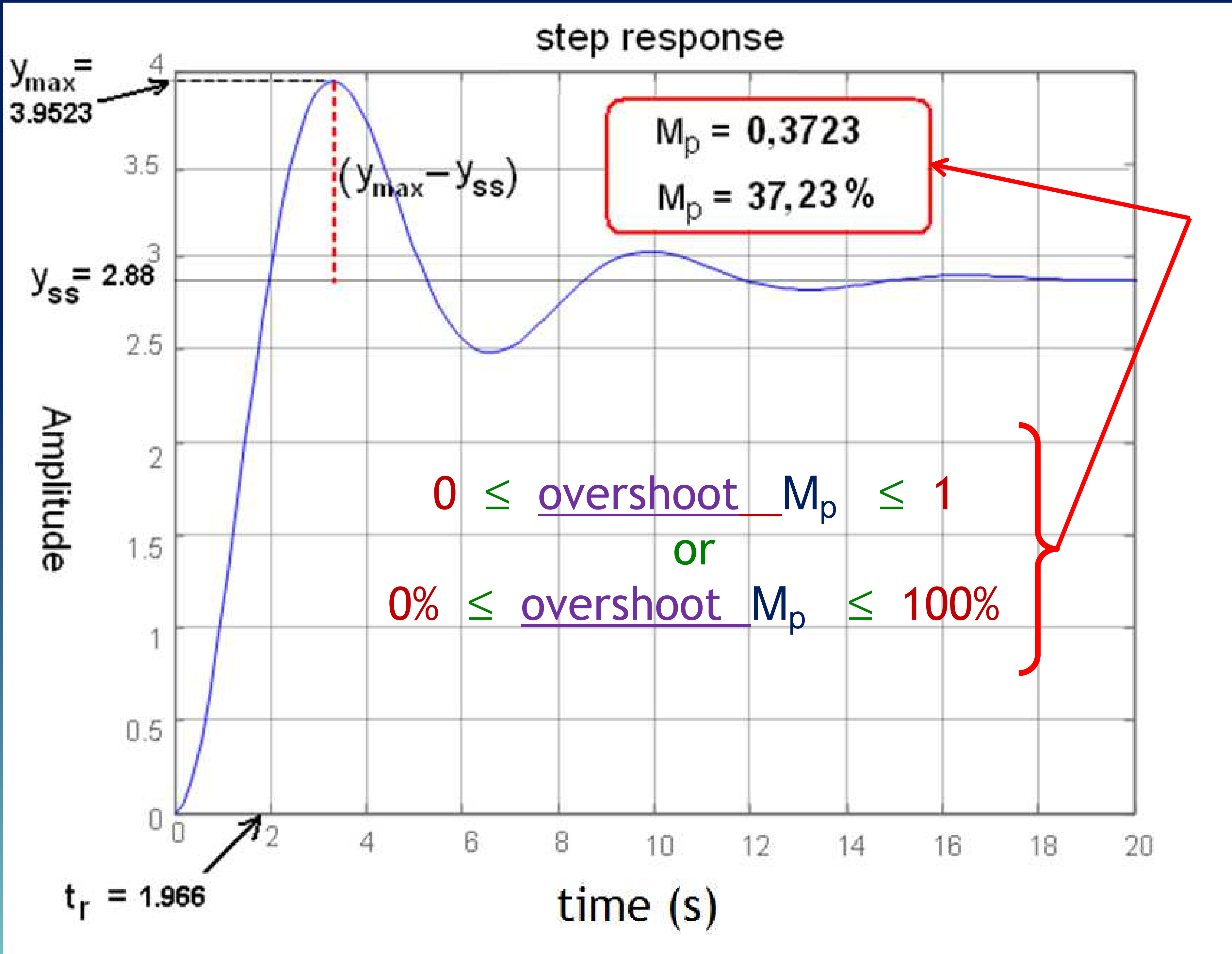
The overshoot M_p can be expressed as a value between 0 and 1.

Time domain analysis - 2nd order systems



Or, instead, it can be expressed as a value between 0% and 100%.

Time domain analysis - 2nd order systems



overshoot

$y_{ss} = K_o$ (*gain*)

ζ (*damping coefficient*),

ω_n (*natural frequency*)

$$M_p = \frac{y_{max} - y_{ss}}{y_{ss}}$$

ou

$$M_p = \frac{y(t_p) - K_o}{K_o}$$

$$M_p = \frac{K_o\left[1 - e^{-\zeta\omega_n t_p}\left(\overbrace{\cos(\pi)}^{-1} - \frac{\zeta}{\sqrt{1-\zeta^2}}\,\overset{0}{\operatorname{sen}(\pi)}\right)\right] - K_o}{K_o}$$

$$M_p = \frac{K_o + K_o e^{-\zeta\omega_n(\pi/\omega_d)} - K_o}{K_o}$$

overshoot

$$M_p = \frac{\cancel{K_o}\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\cancel{K_o}}$$

$$\Rightarrow \quad M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

M_p depends only on ζ (*damping coefficient*)

overshoot

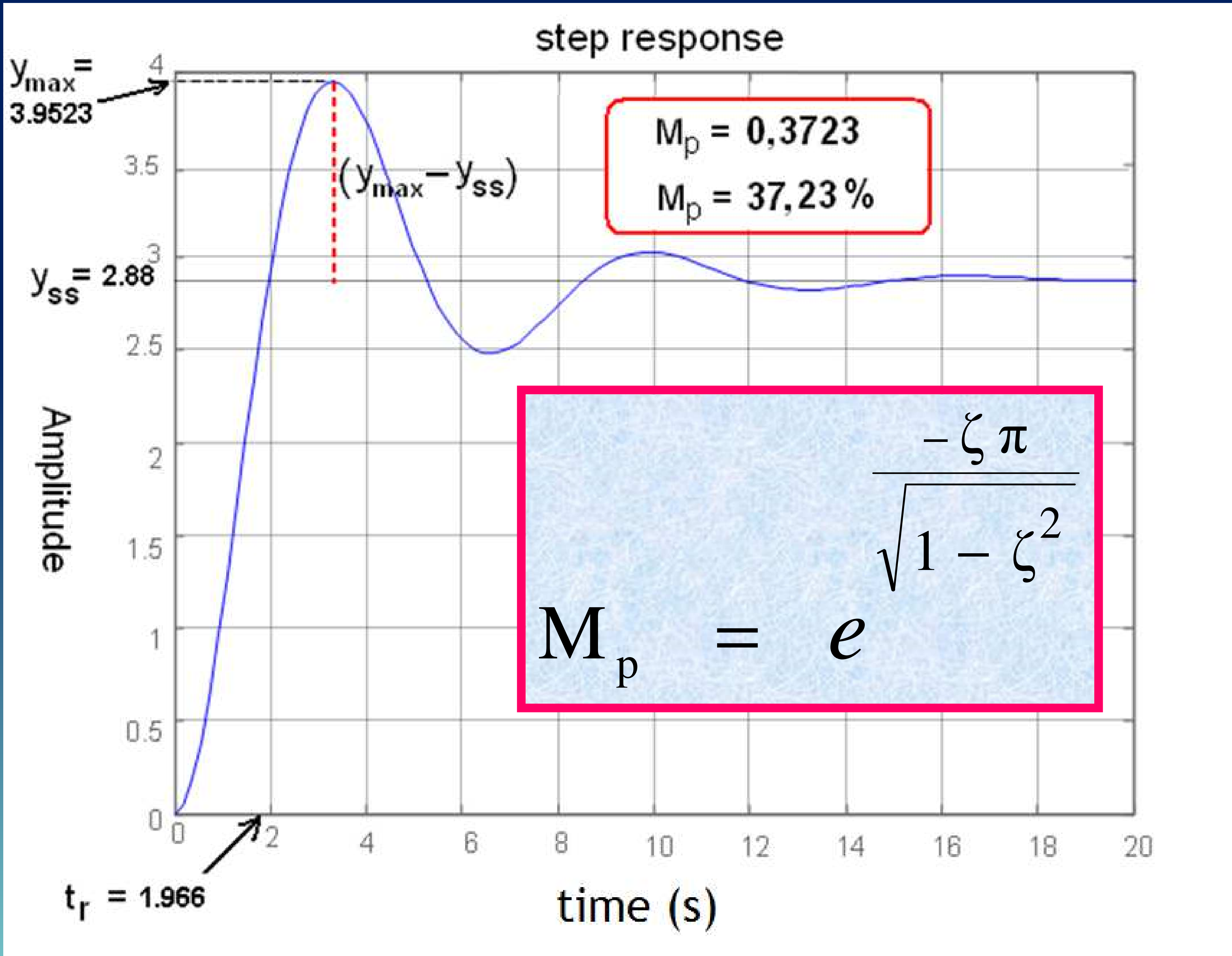
M_p depends only on ζ (*damping coefficient*)

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

or

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\,\%$$

Time domain analysis – 2^{nd} order systems

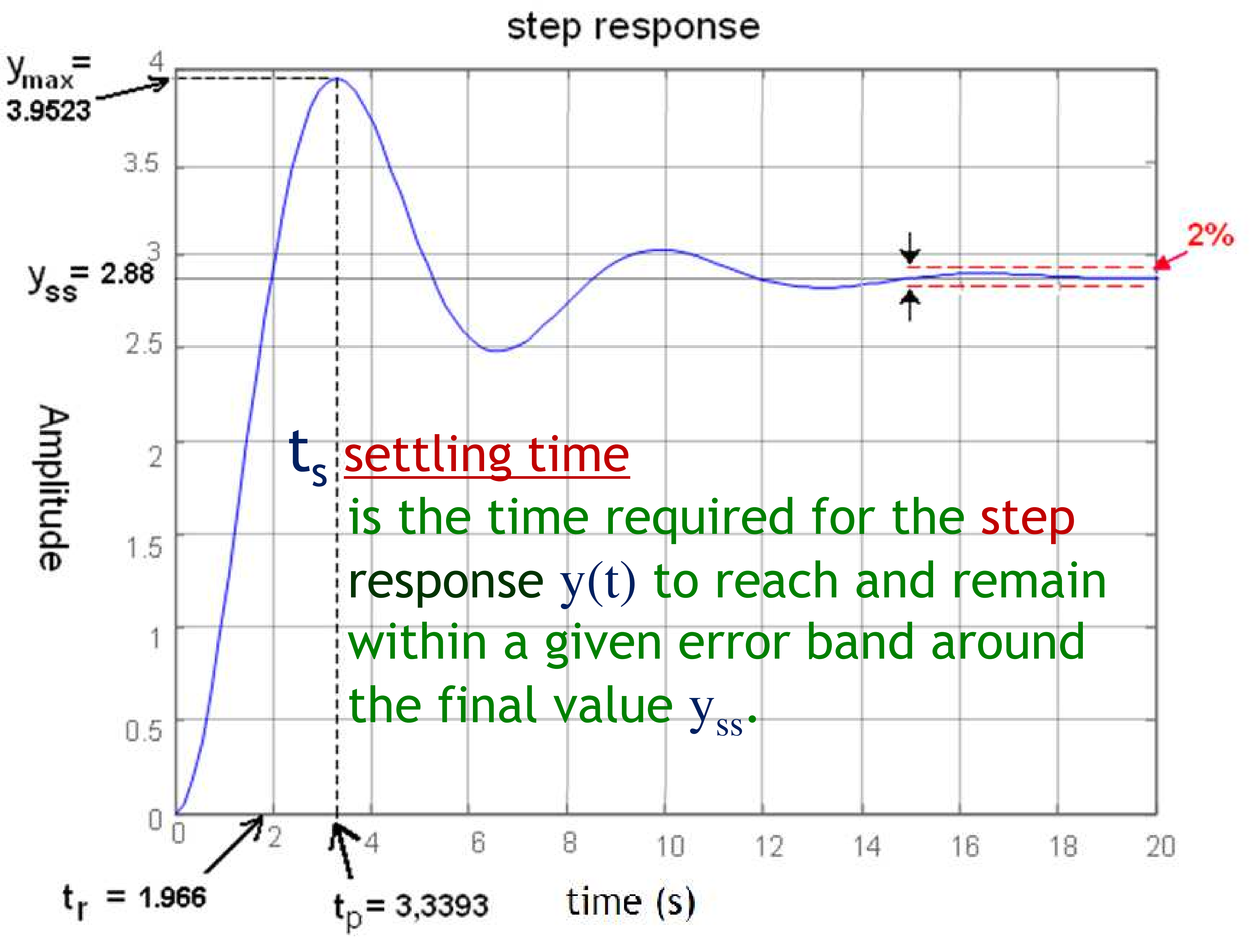


$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

settling time

t_s

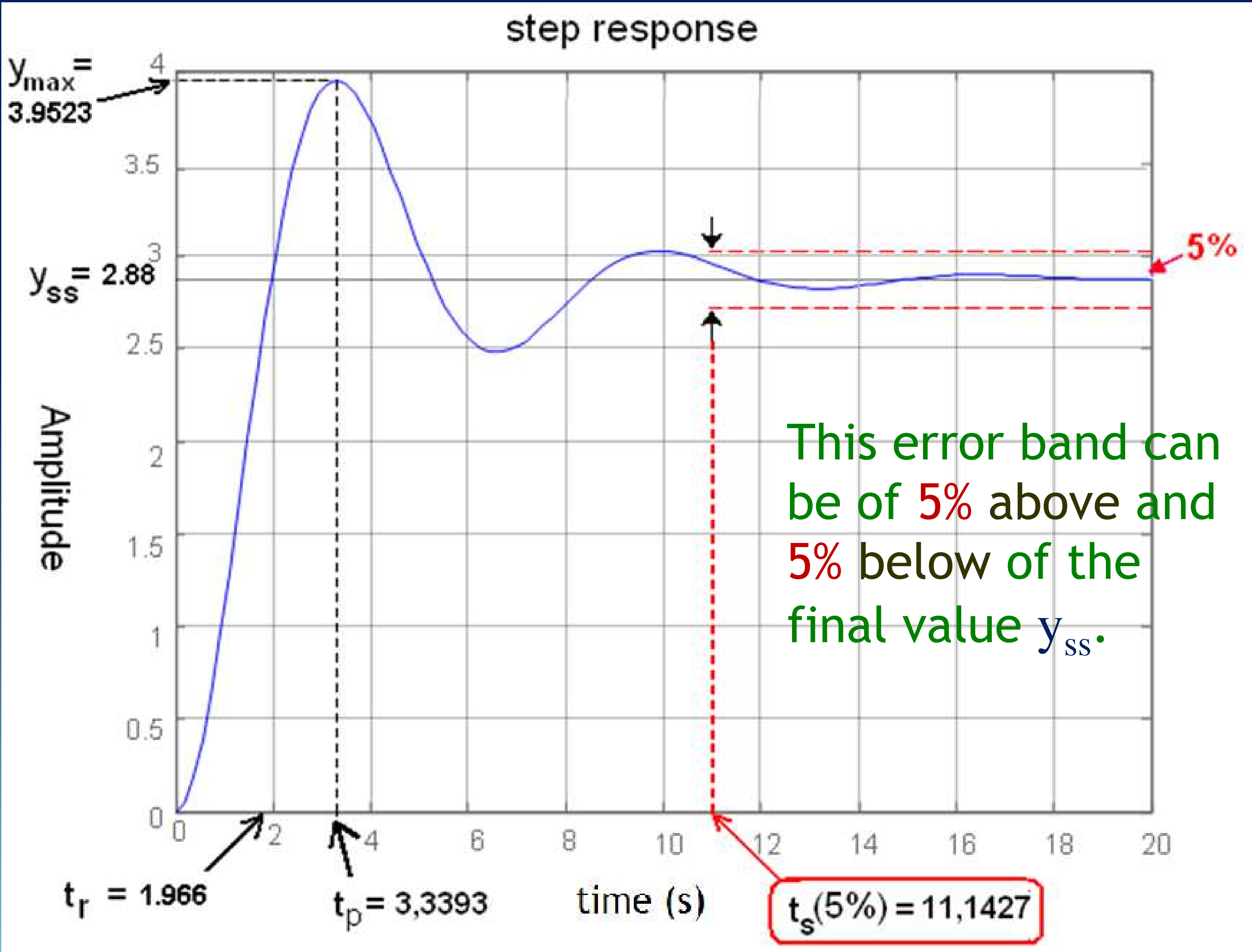
Time domain analysis - 2nd order systems



t_s settling time

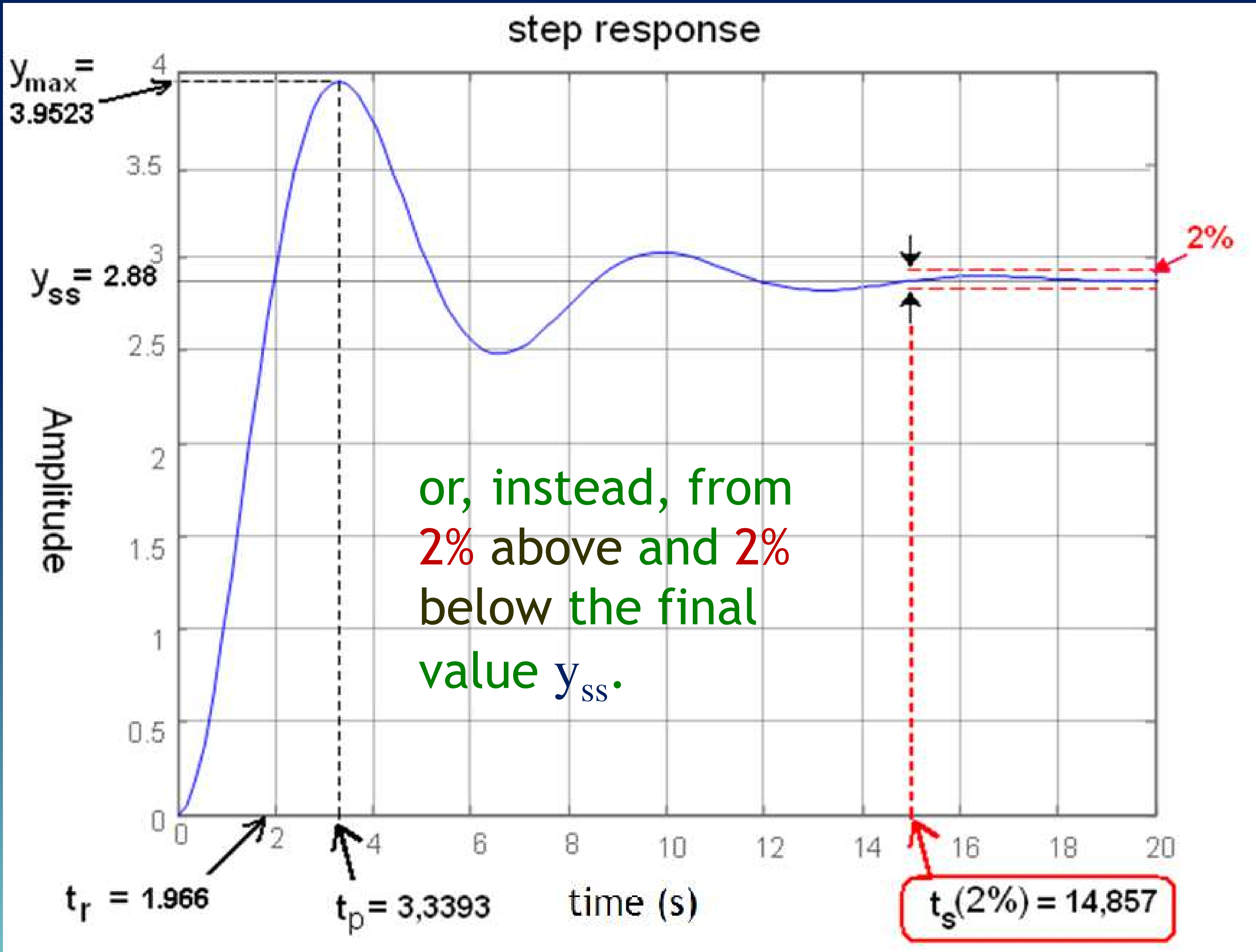
is the time required for the step response $y(t)$ to reach and remain within a given error band around the final value y_{ss}.

Time domain analysis - 2nd order systems

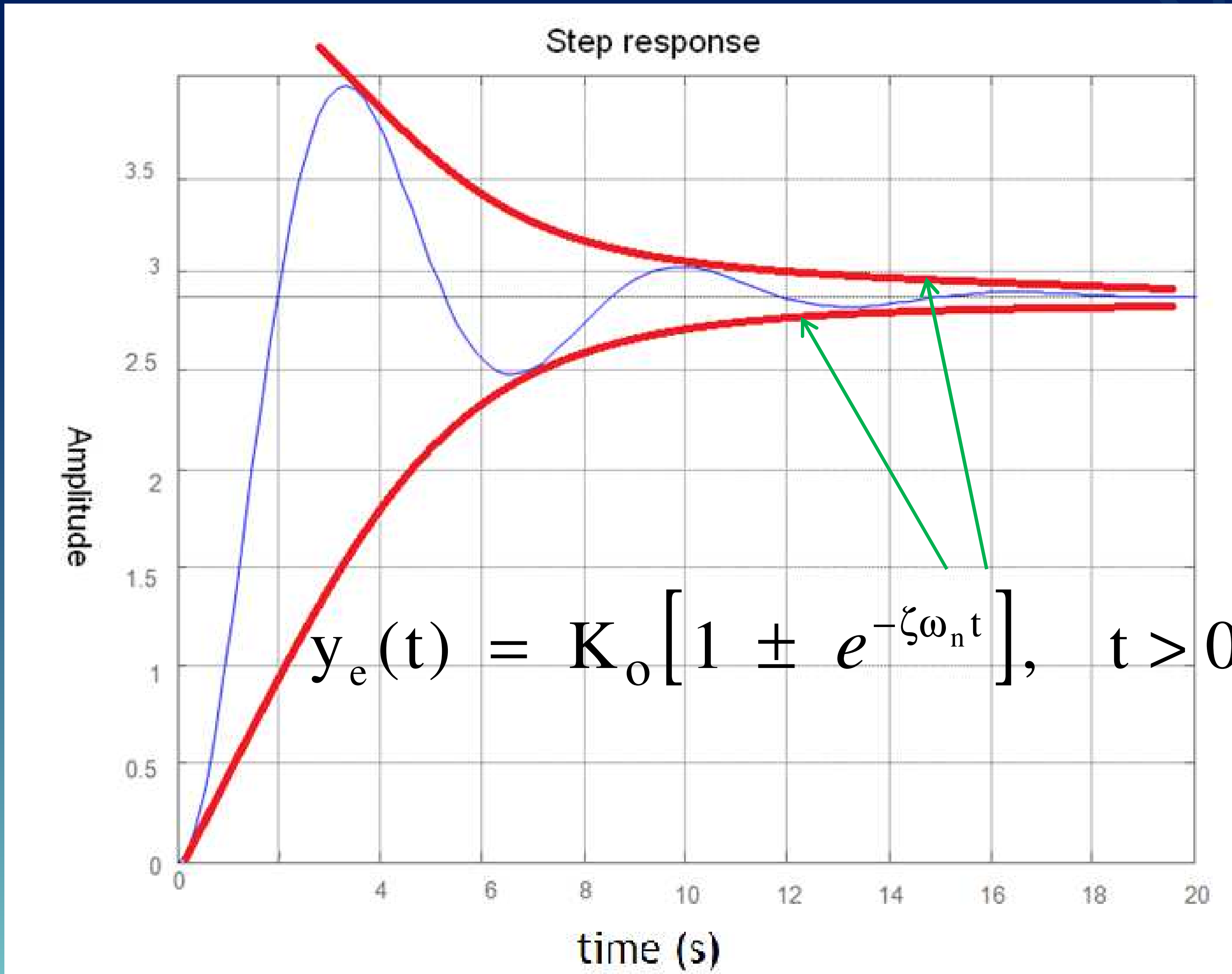


This error band can be of 5% above and 5% below of the final value y_{ss}.

Time domain analysis - 2nd order systems



The settling time is obtained from the equations of $y_e(t)$, the curves that encompass $y(t)$.



$$y_e(t) = K_o\left[1 \pm e^{-\zeta\omega_n t}\right], \quad t > 0$$

That is, the *settling time* t_s is obtained by calculating

$$y_e(t_s) \approx 1{,}05\ K_o$$

for the case of t_s with 5% tolerance, and

$$y_e(t_s) \approx 1{,}02\ K_o$$

for the case of t_s with 2% tolerance,

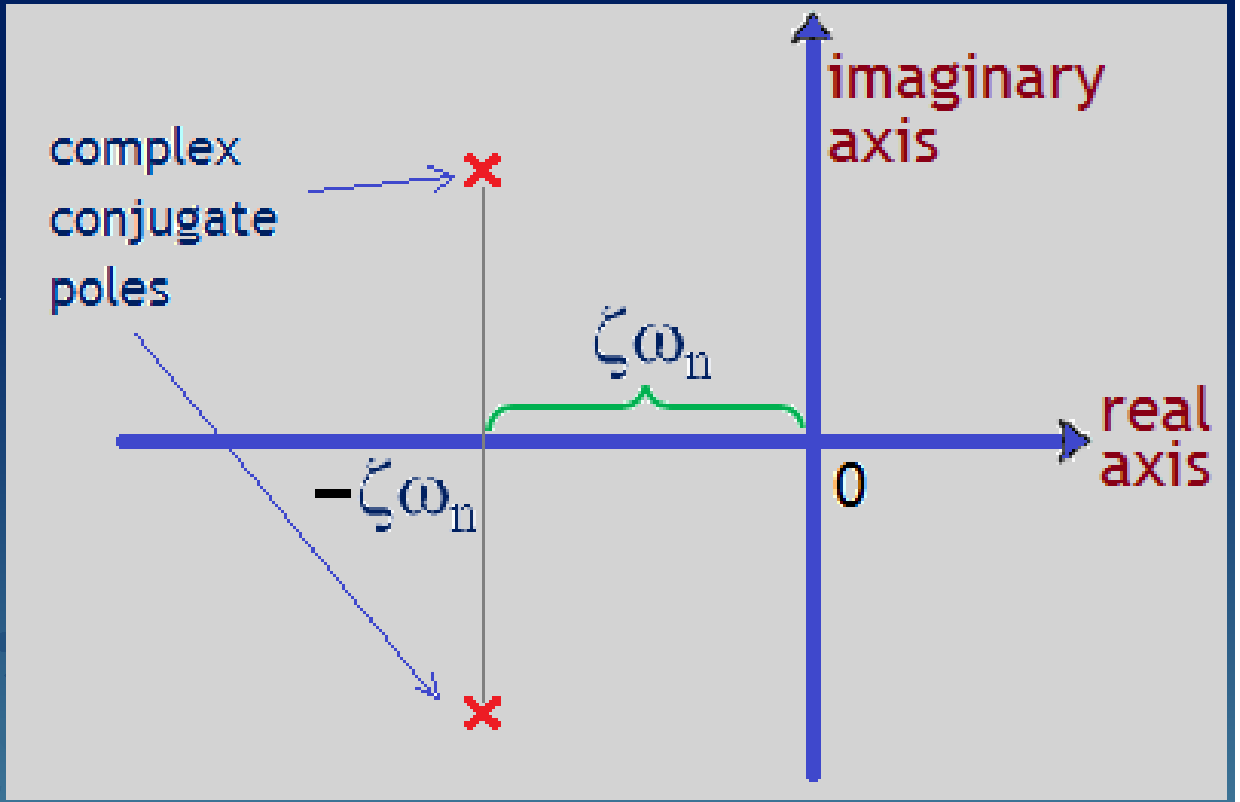
obtaining the following values:

$$t_s(5\%) = \frac{3}{\zeta\omega_n}$$

$$t_s(2\%) = \frac{4}{\zeta\omega_n}$$

settling time

Note that the settling time t_r is inversely proportional to $\zeta\omega_n$, which is the distance of the real part of the poles to the origin.



hence:

$$t_s(5\%) = \frac{3}{\zeta\omega_n}$$

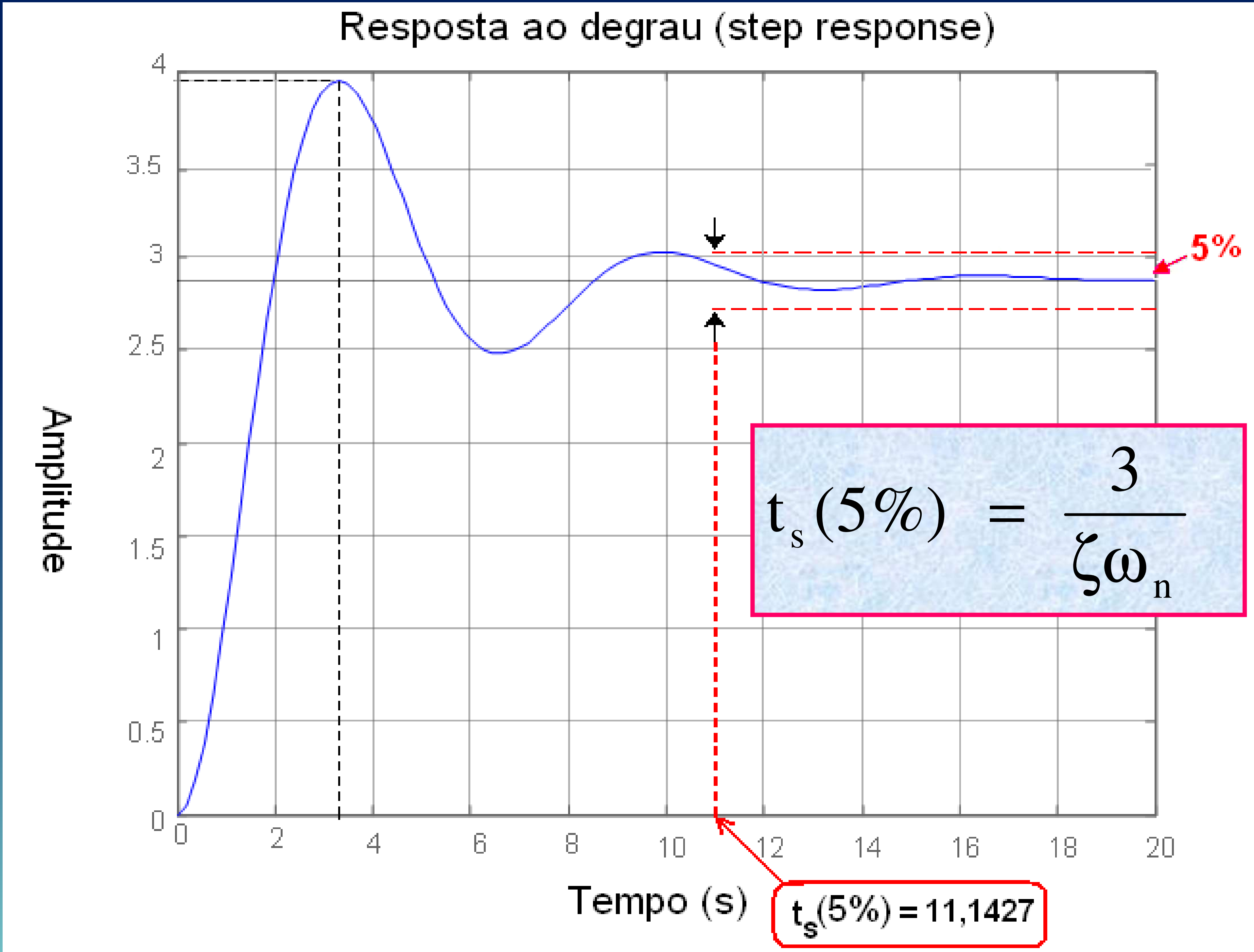
and

$$t_s(2\%) = \frac{4}{\zeta\omega_n}$$

$t_s(5\%) = 3 / \zeta\omega_n$

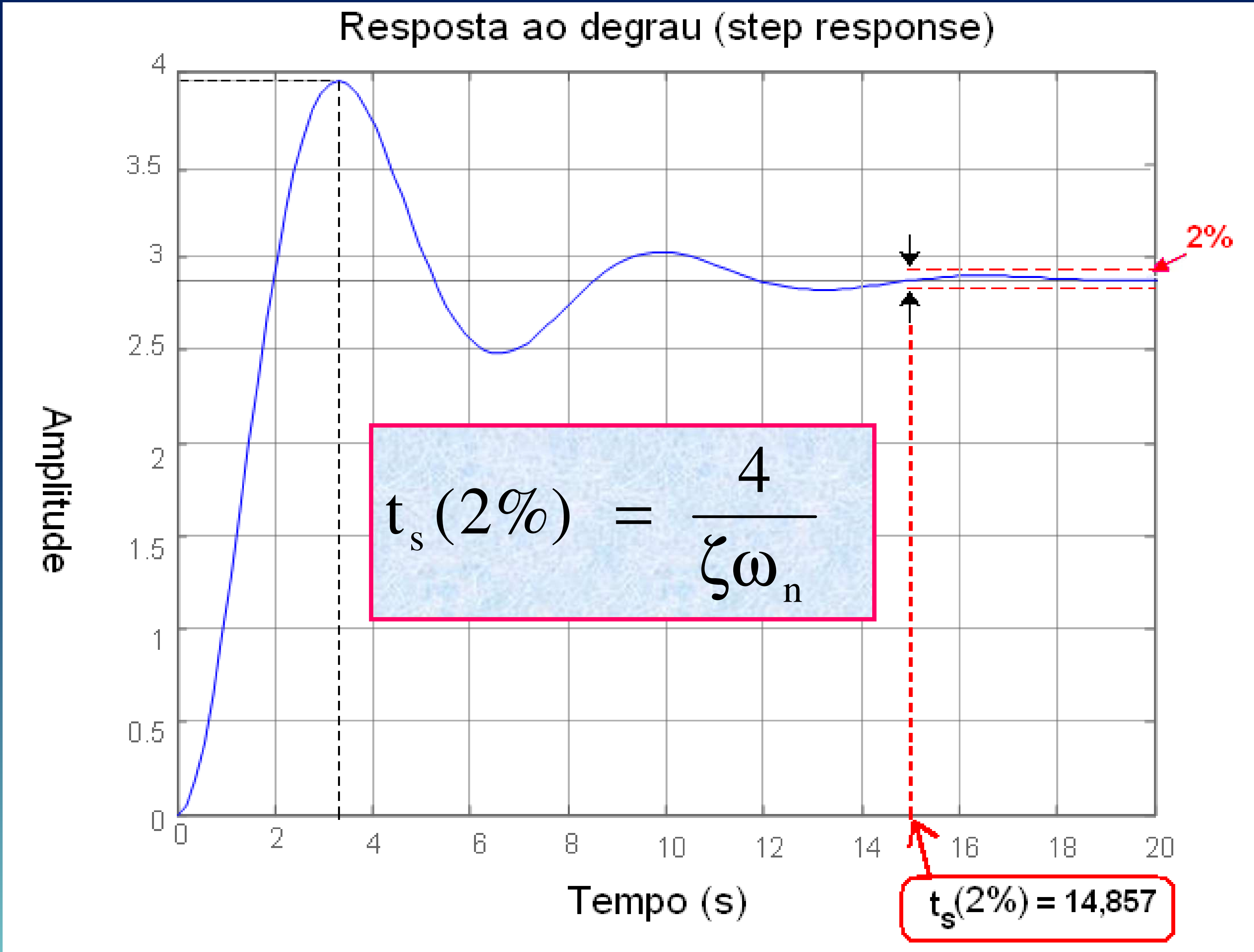
$t_s(2\%) = 4 / \zeta\omega_n$

Time domain analysis - 2nd order systems



$$t_s(5\%) = \frac{3}{\zeta \omega_n}$$

Time domain analysis - 2nd order systems



$$t_s(2\%) = \frac{4}{\zeta\omega_n}$$

Note that we have seen cases in which

$$0 < \zeta < 1$$

$$\zeta = 1$$

$$\zeta > 1$$

that is:

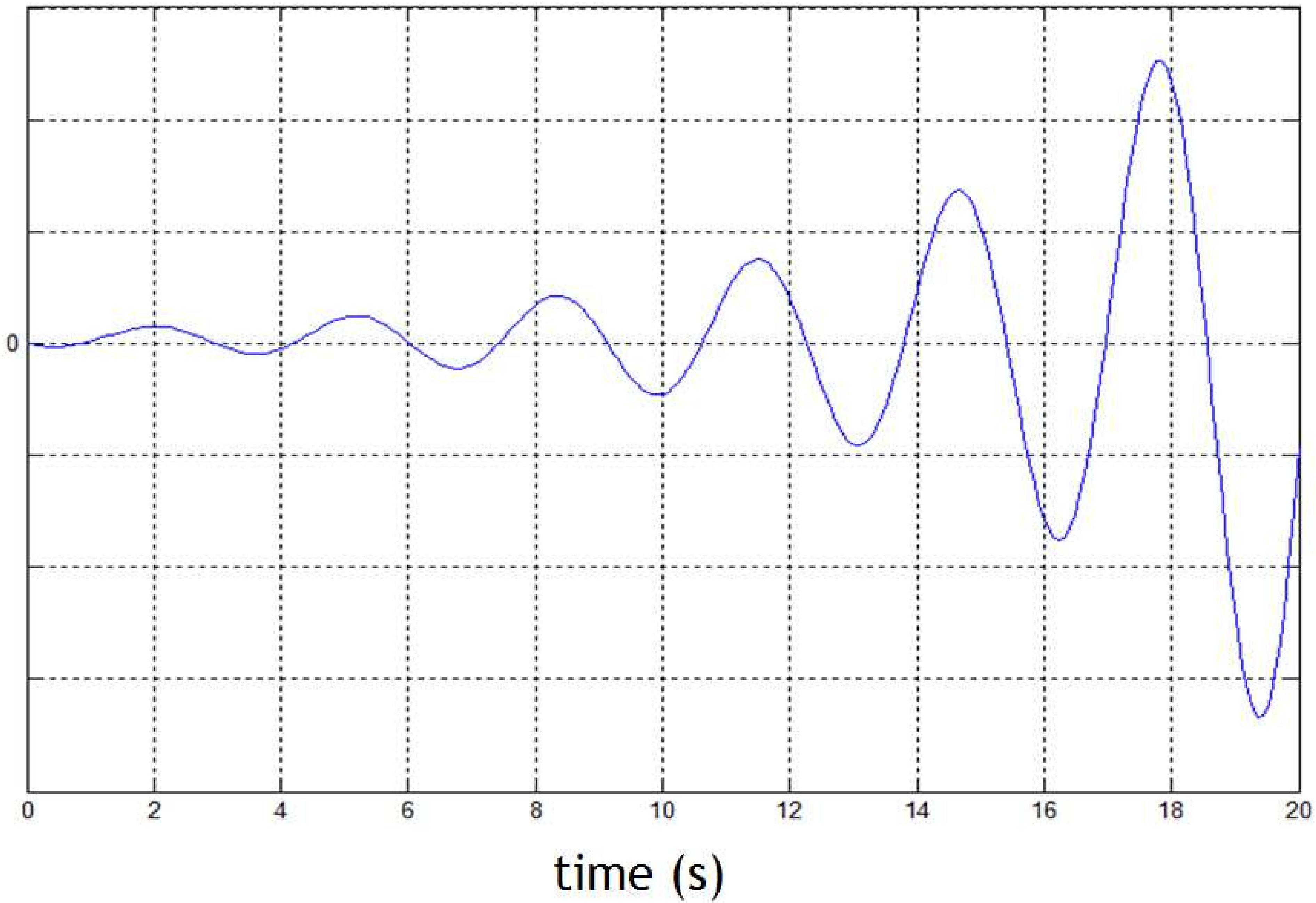
$$\zeta > 0$$

However, if $\zeta < 0$ then:

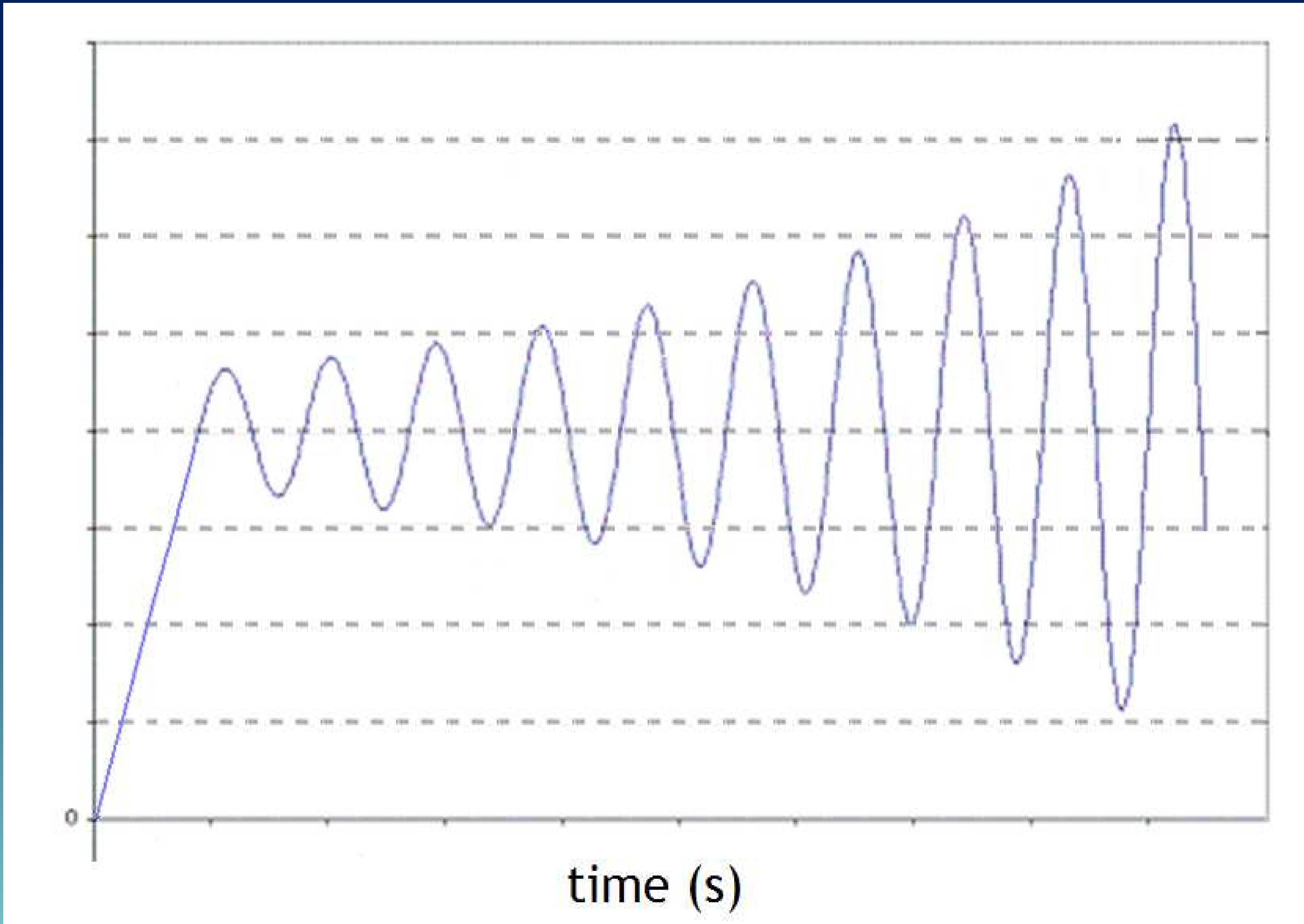
the system is unstable

Time domain analysis - 2nd order systems

$\zeta < 0 \rightarrow$ unstable system (an example)



$\zeta < 0 \rightarrow$ unstable system (another example)



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Thank you!

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