Control Systems

4

"Systems Representation"

J. A. M. Felippe de Souza

Systems Representation

input

System

output

Transfer Function

Relation between:

$Y(s)$

$$\frac{\text{Laplace Transform of the output } y(t)}{\text{Laplace Transform of the input } x(t)}$$

$X(s)$

considering inicial conditions zero.

Systems Representation

Pierre Simon Laplace,
1749-1827

$$\text{Transfer Function} = \frac{Y(s)}{X(s)}$$

$X(s)$ = Laplace Transform of $x(t)$

$Y(s)$ = Laplace Transform of $y(t)$

Systems Representation

$$\text{Transfer Function} = \frac{Y(s)}{X(s)}$$

output

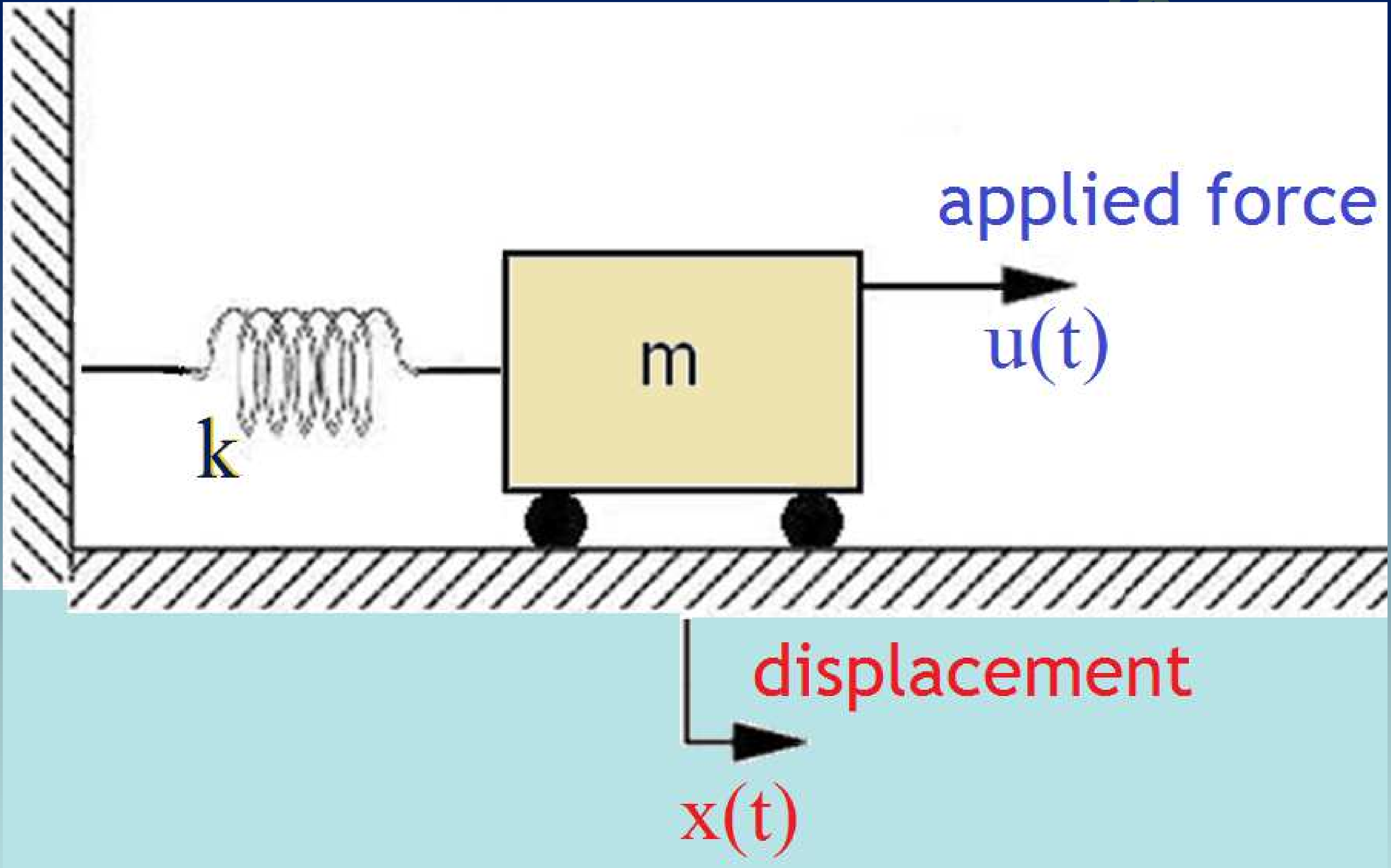
input

X(s) = Laplace Transform of x(t)

Y(s) = Laplace Transform of y(t)

cart / mass / spring

cart / mass / spring



Systems Representation

cart / mass / spring

entrada
(input)

saída
(output)

$$\text{F.T.} = \frac{X(s)}{U(s)}$$

output

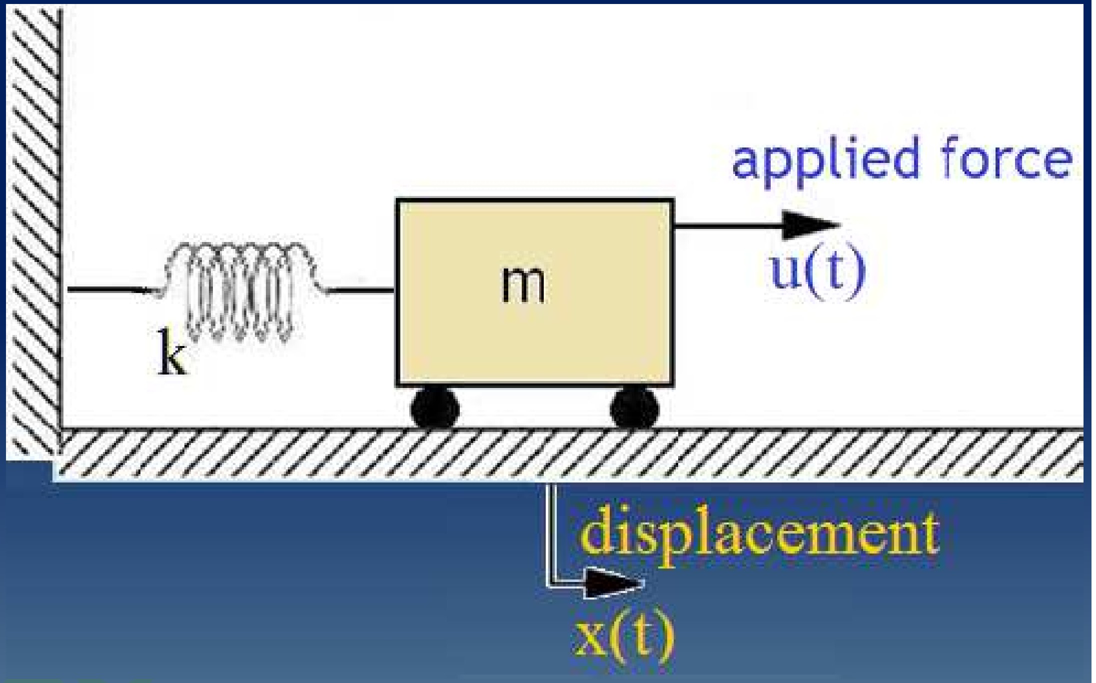
input

$U(s)$ = Laplace Transform of $u(t)$

$X(s)$ = Laplace Transform of $x(t)$

Systems Representation

cart / mass / spring

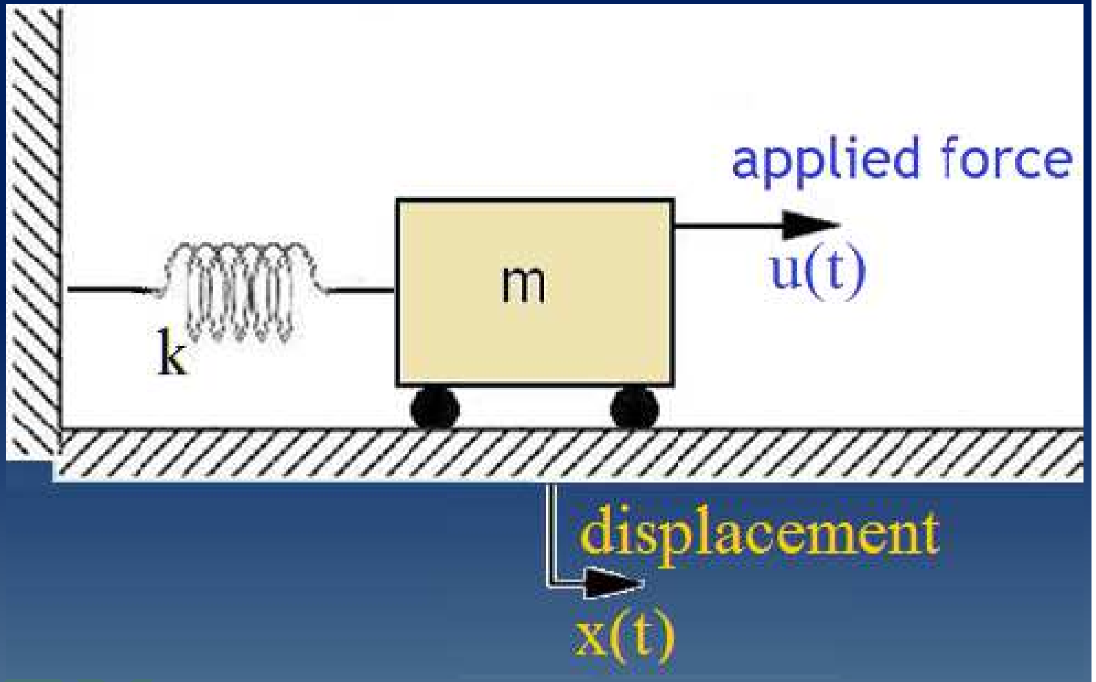


$$mx'' + \mu x' + kx = u,$$

or

$$m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + k\,x = u,$$

cart / mass / spring

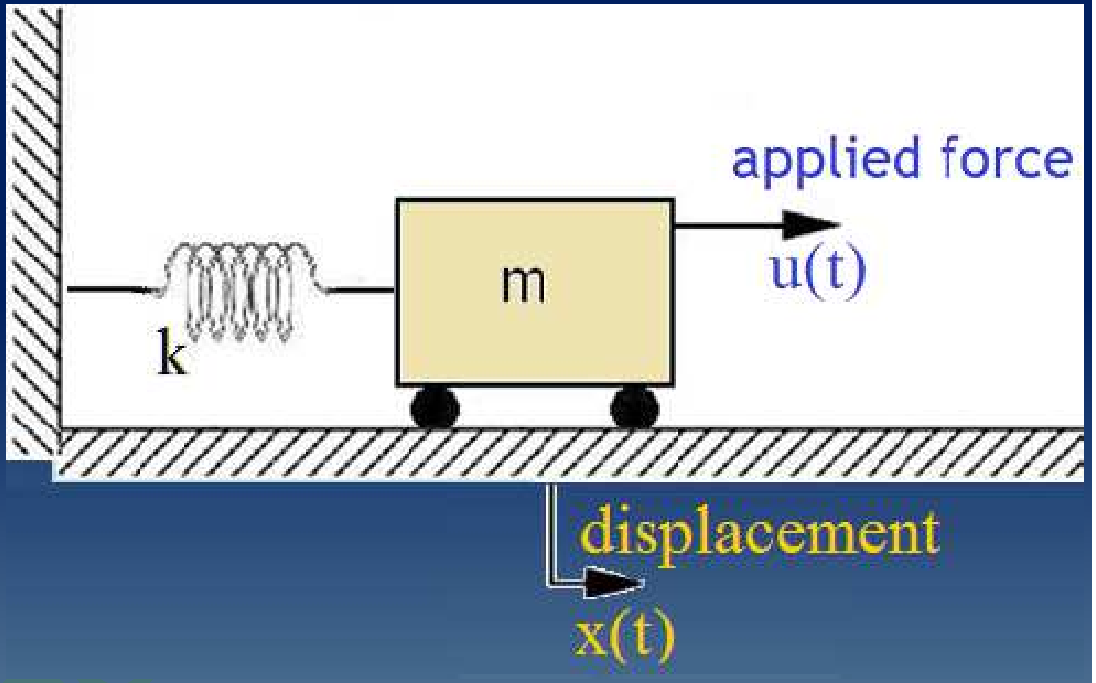


$$\begin{cases} m\dfrac{d^2x}{dt^2} + \mu\dfrac{dx}{dt} + k\,x = mx'' + \mu x' + kx = u, \\ x'(0) = 0, \qquad x(0) = 0 \end{cases}$$

hence,

$$m\,s^2X(s) + \mu\,s\,X(s) + k\,X(s) \;=\; U(s),$$

cart / mass / spring

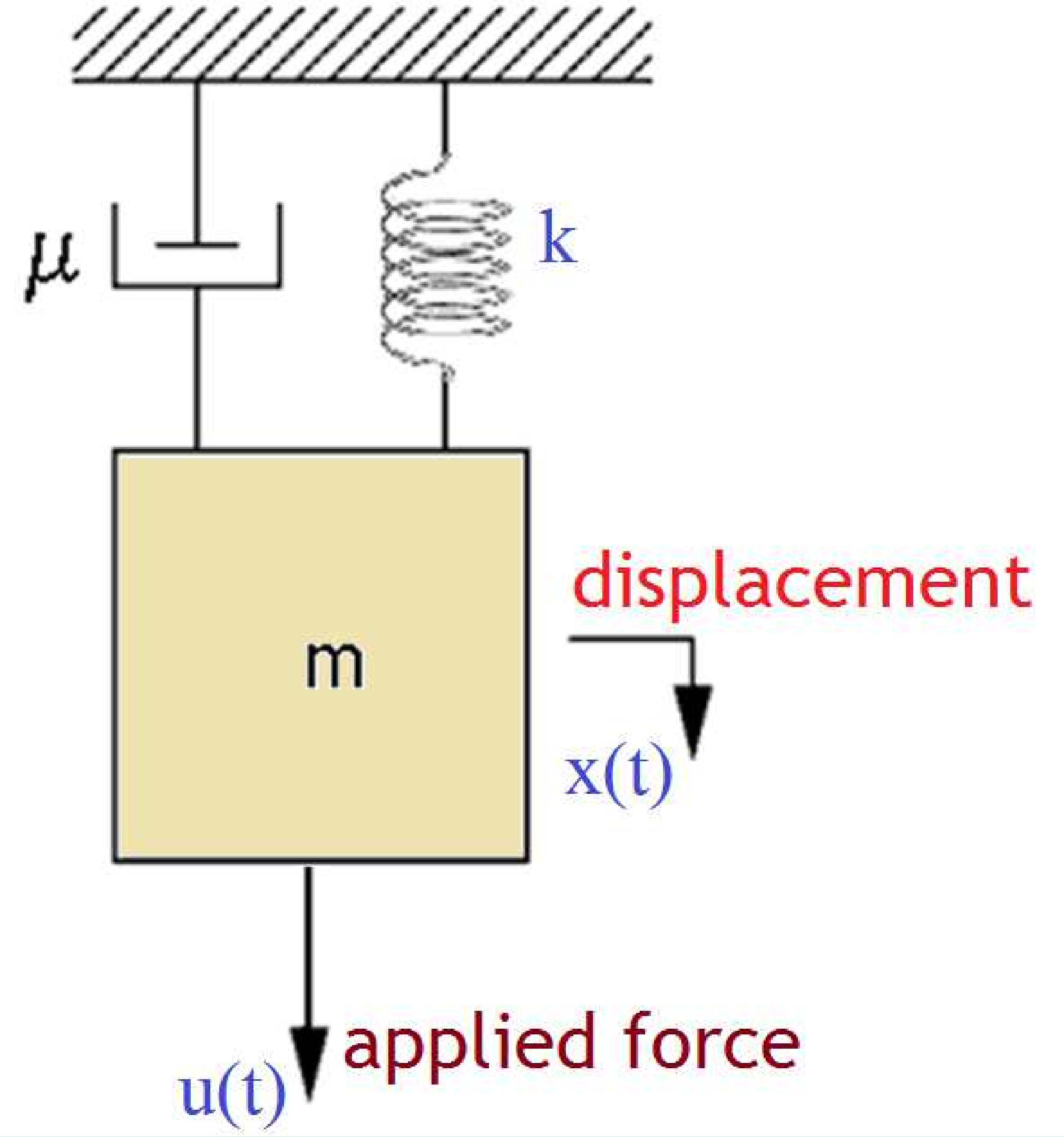


and then, the Transfer Function (T. F.) is given by

$$F.T. = \frac{X(s)}{U(s)} = \frac{1}{ms^2 + \mu s + k}$$

translational mechanical movement

translational
mechanical
movement



Systems Representation

translational mechanical movement

u(t)

S

x(t)

input

output

$$\boxed{\text{F.T.} = \frac{X(s)}{U(s)}}$$

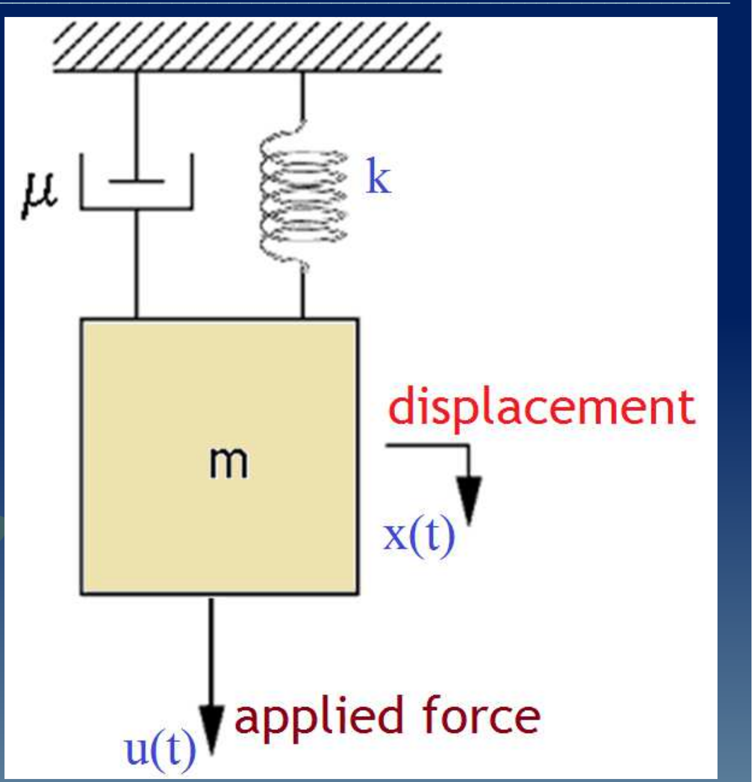
output

input

$U(s)$ = Laplace Transform of $u(t)$

$X(s)$ = Laplace Transform of $x(t)$

translational mechanical movement

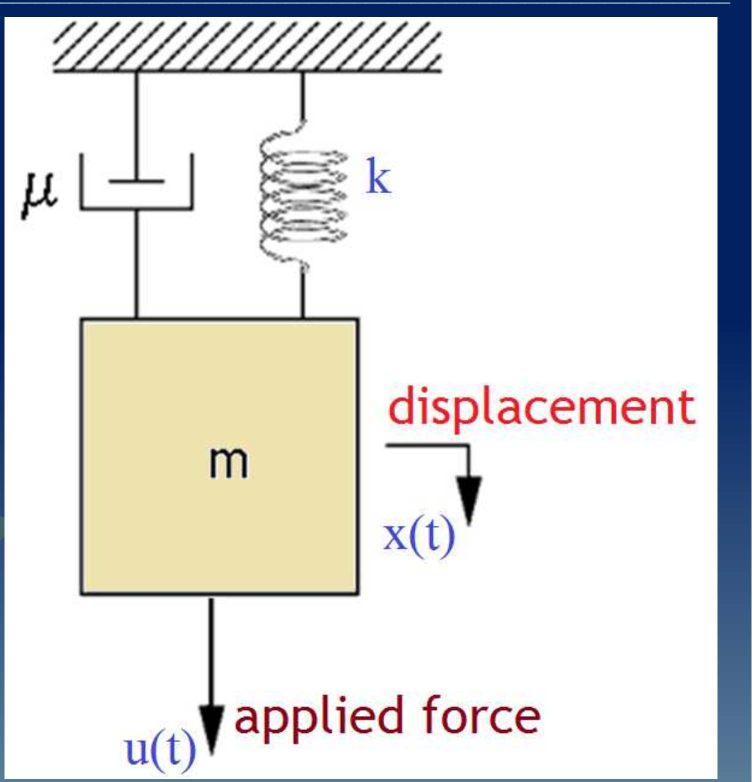


$$mx'' + \mu x' + kx = u,$$

or

$$m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + k\,x = u,$$

translational mechanical movement



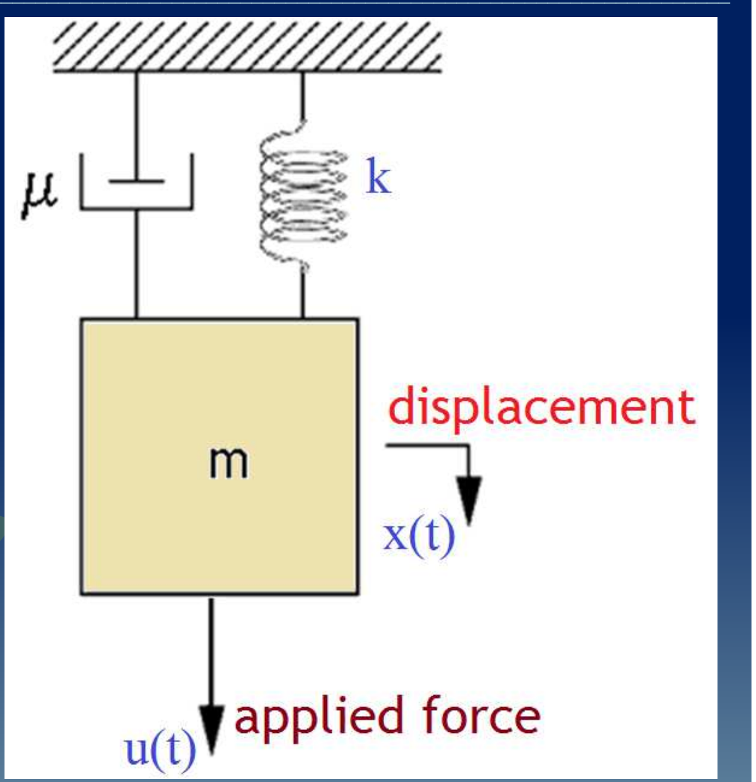
$$\begin{cases} m\dfrac{d^2x}{dt^2} + \mu\dfrac{dx}{dt} + k\,x = mx'' + \mu x' + kx = u, \\ x'(0) = 0, \qquad x(0) = 0 \end{cases}$$

thus,

$$m\,s^2X(s) + \mu\,s\,X(s) + k\,X(s) \;=\; U(s),$$

Systems Representation

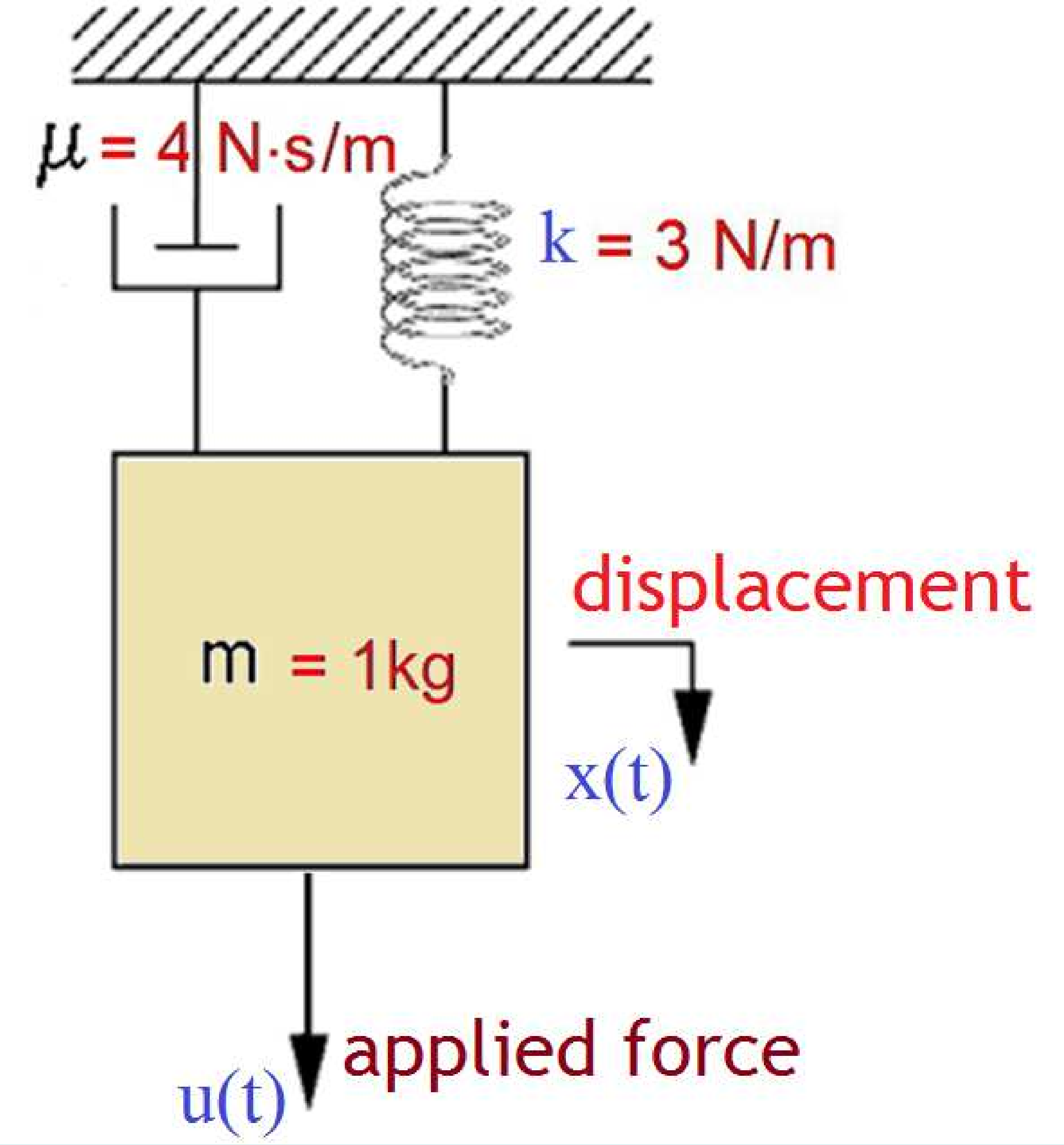
translational mechanical movement



and then, the Transfer Function (T. F.) becomes

$$\text{F.T.} = \frac{X(s)}{U(s)} = \frac{1}{ms^2 + \mu s + k}$$

translational mechanical movement

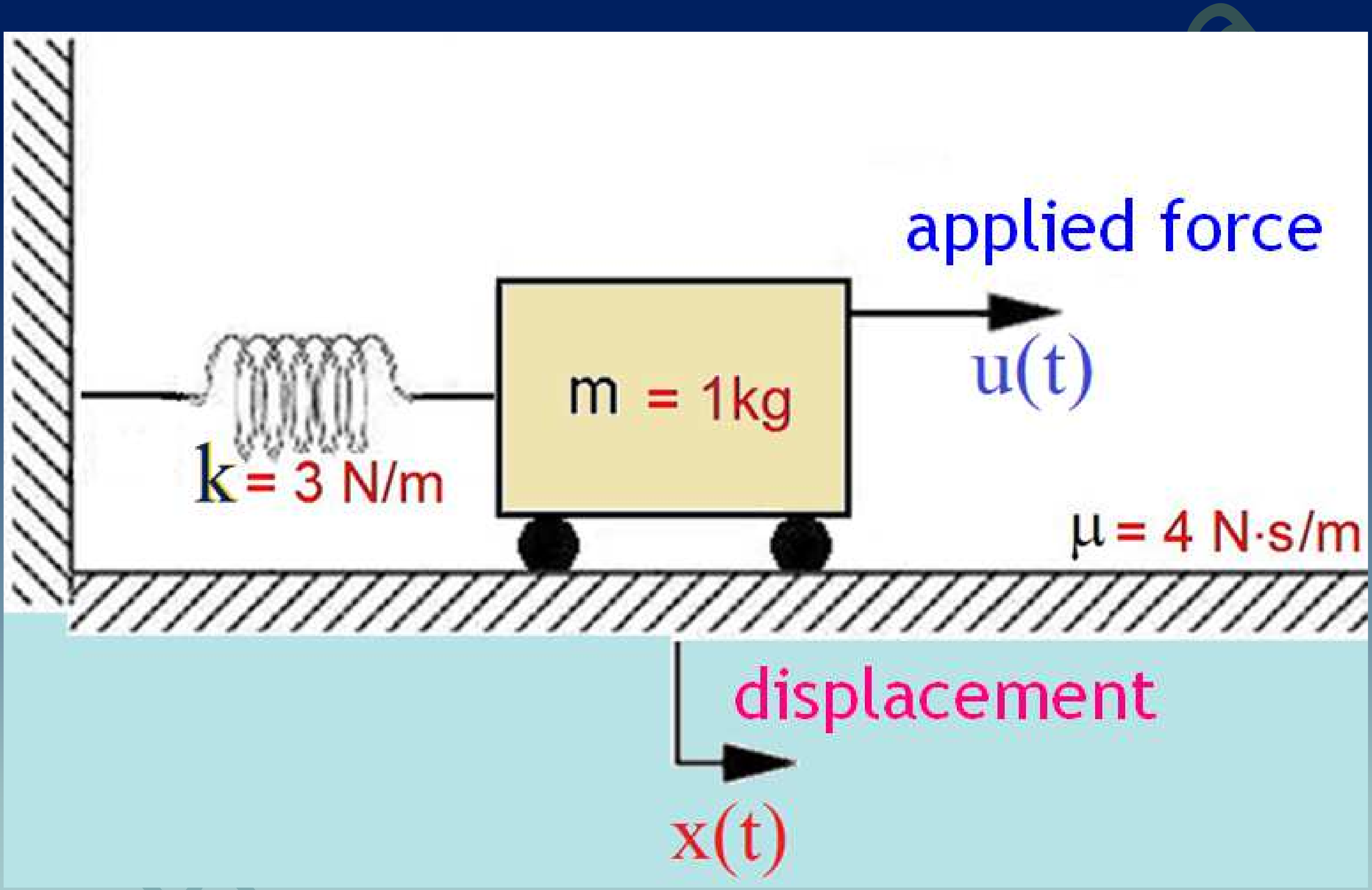


m = 1 kg

μ = 4 N·s/m

k = 3 N/m

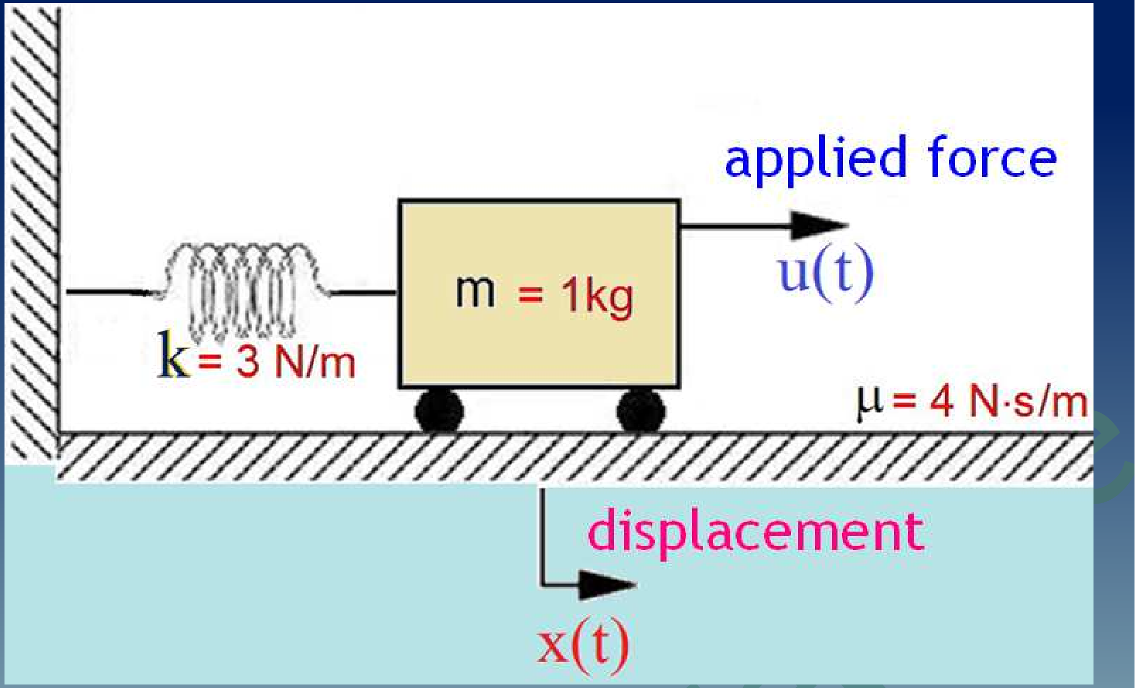
cart / mass / spring

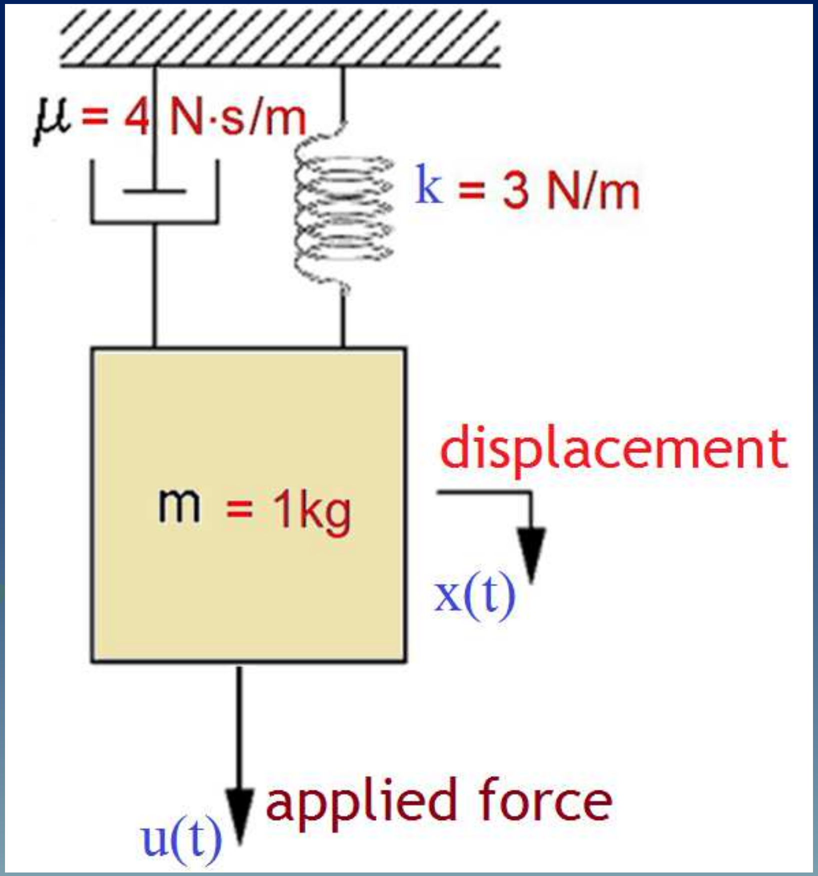


m = 1 kg μ = 4 N·s/m k = 3 N/m

Systems Representation

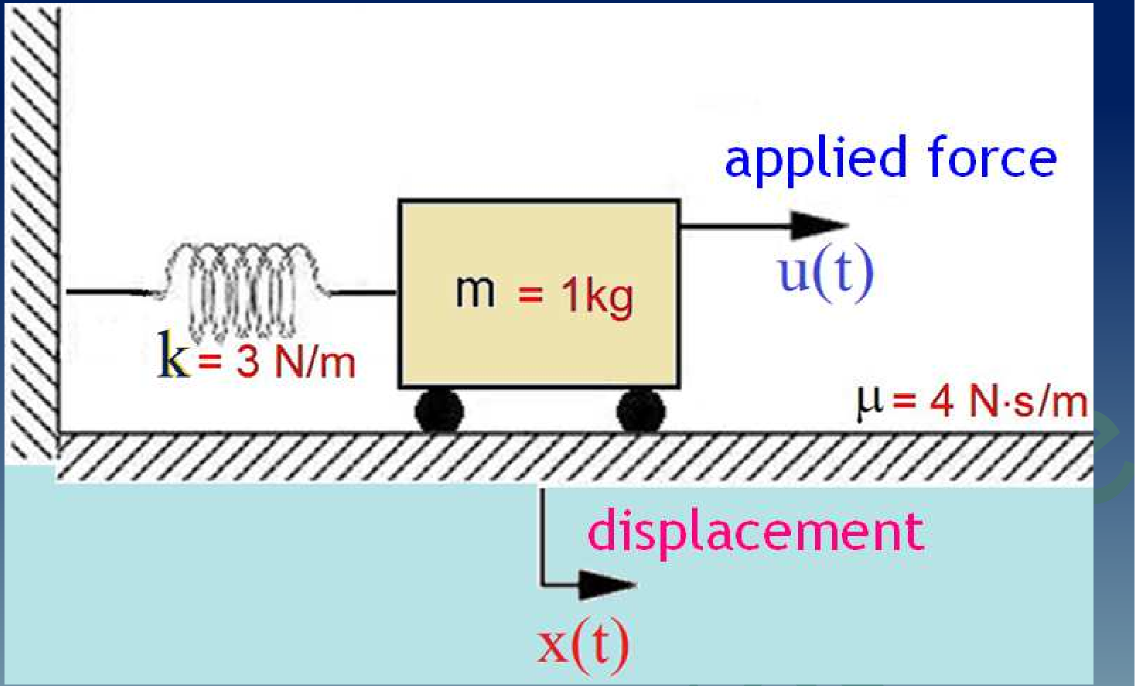
cart / mass / spring or translational mechanical movement

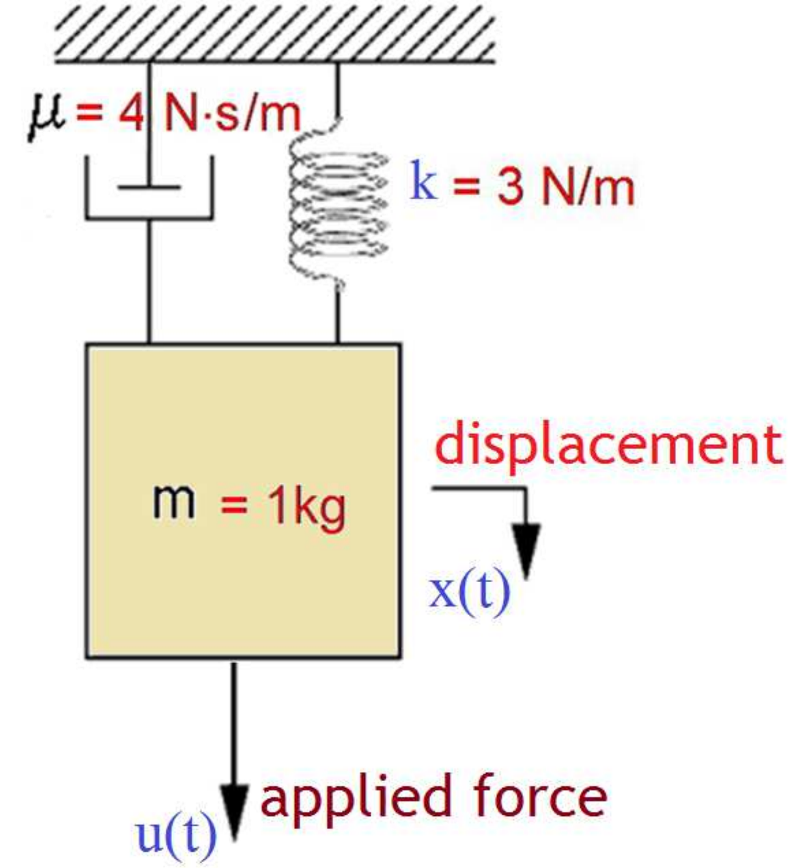




We have seen that these 2 systems are described by the same *differential equation* (of 2nd order) and have the same *model*.

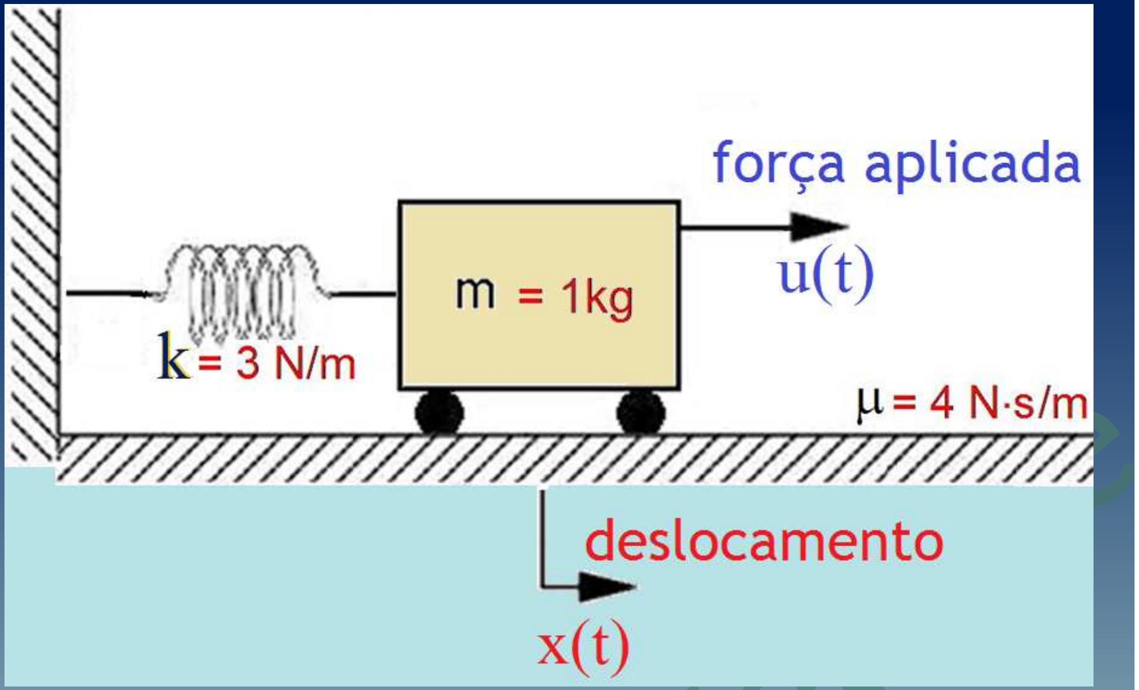
cart / mass / spring or translational mechanical movement

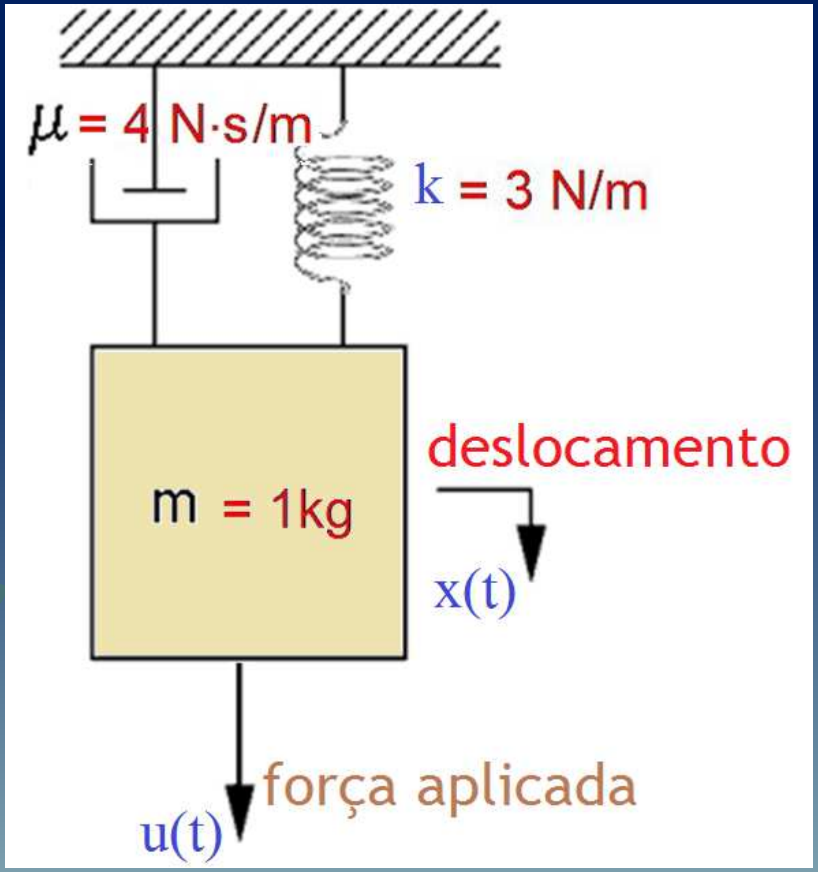




$$\begin{cases} \dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = x'' + 4x' + 3x = u, \\ x'(0) = 0, \qquad x(0) = 0 \end{cases}$$

cart / mass / spring or translational mechanical movement



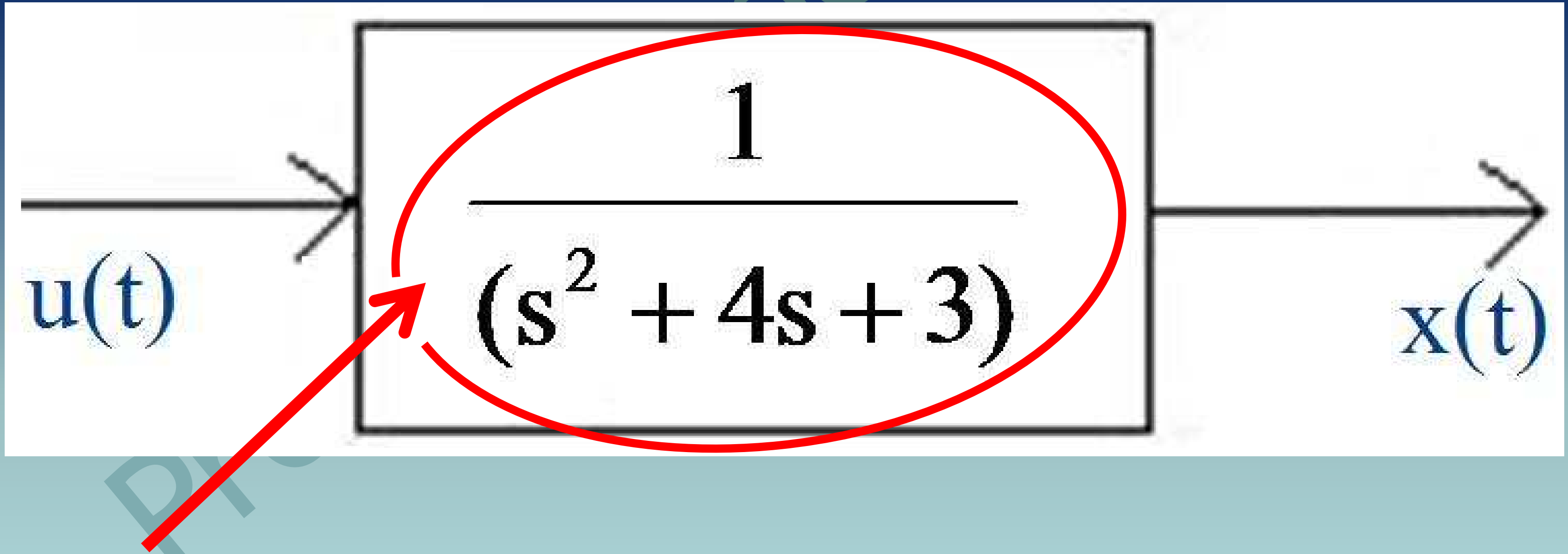


Hence, the Transfer Function (T.F.) is

$$\text{F.T.} = \frac{X(s)}{U(s)} = \frac{1}{s^2 + 4s + 3}$$

Systems Representation

cart / mass / spring or translational mechanical movement

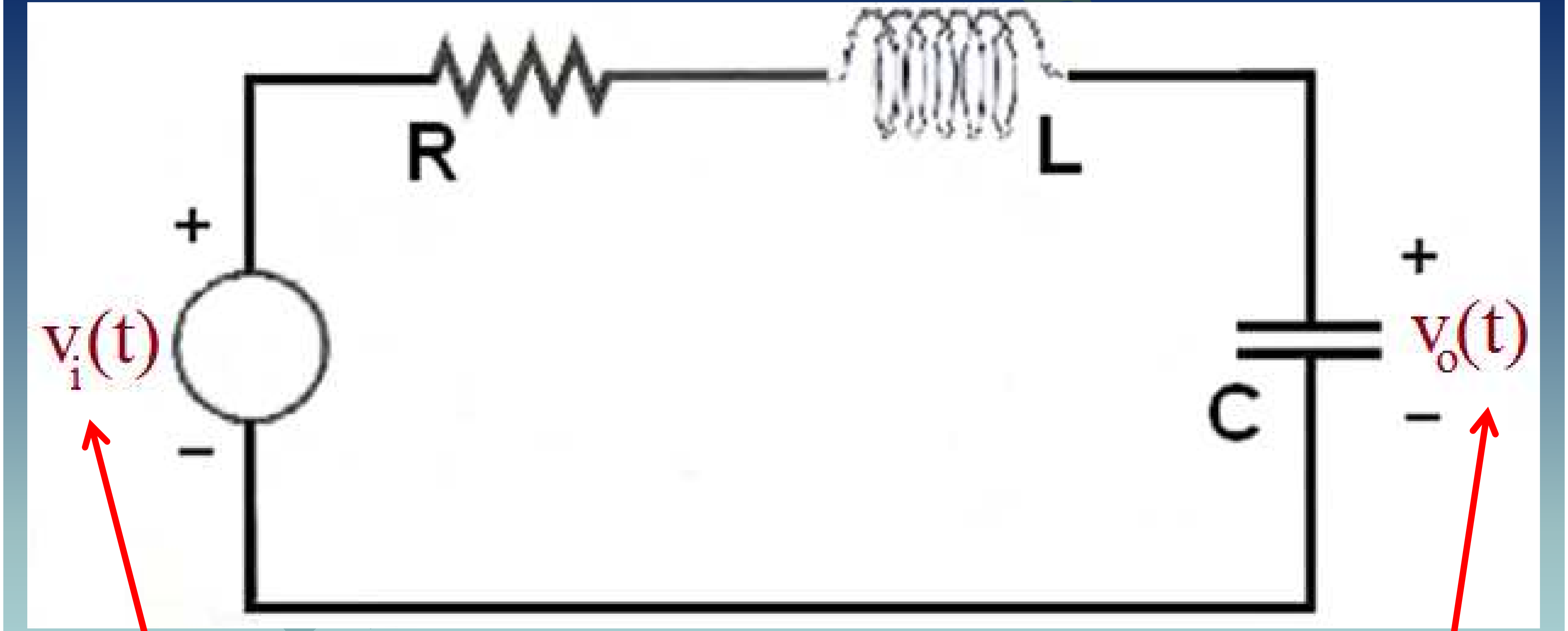


$$\frac{1}{(s^2 + 4s + 3)}$$

Transfer Function (T.F.)
of the system

RLC circuit series

RLC circuit series



tensão
na entrada

tensão
na saída

Systems Representation

RLC circuit series

output

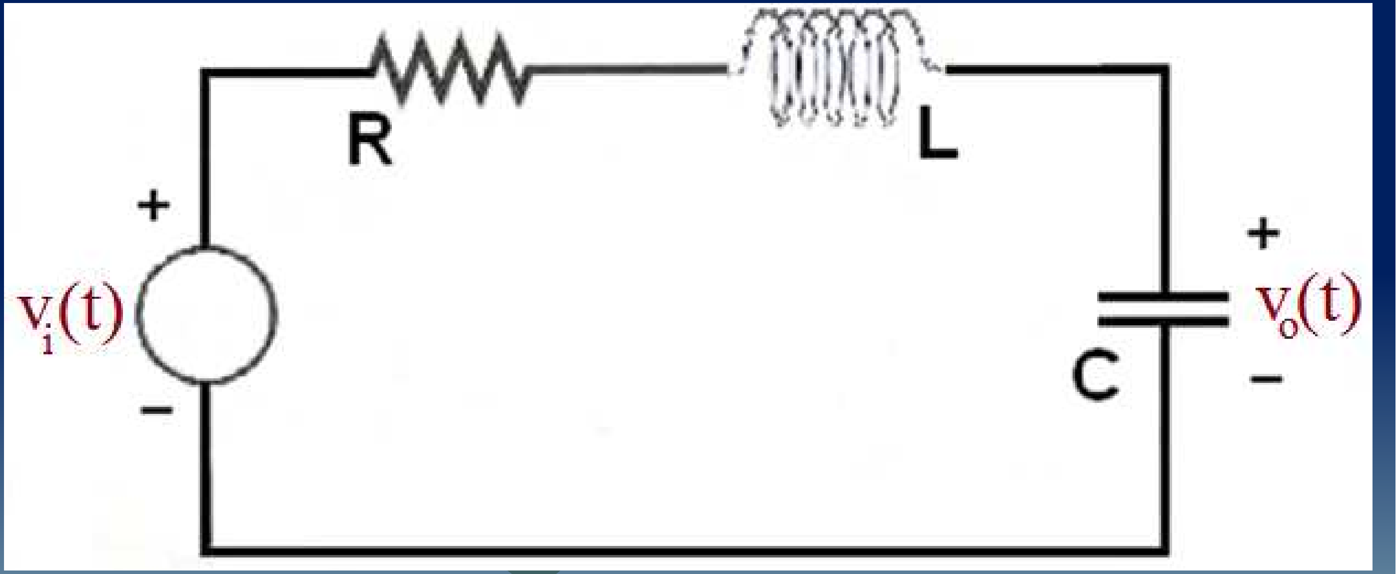
$$\boxed{\text{F.T.} = \frac{V_o(s)}{V_i(s)}}$$

input

$V_i(s)$ = Laplace Transform of $v_i(t)$

$V_o(s)$ = Laplace Transform of $v_o(t)$

RLC circuit series

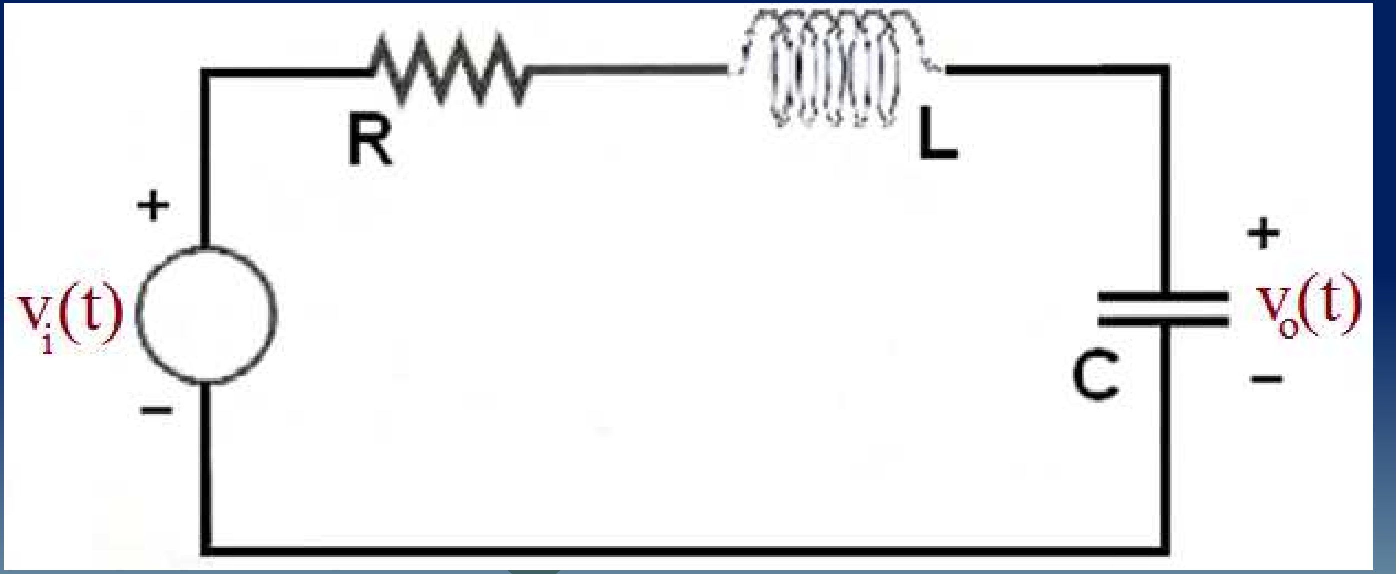


$$LC\, v_o'' + RC\, v_o' + v_o = v_i \,,$$

or

$$LC\frac{d^2 v_o}{dt^2} + RC\frac{dv_o}{dt} + v_o = v_i \,,$$

RLC circuit series

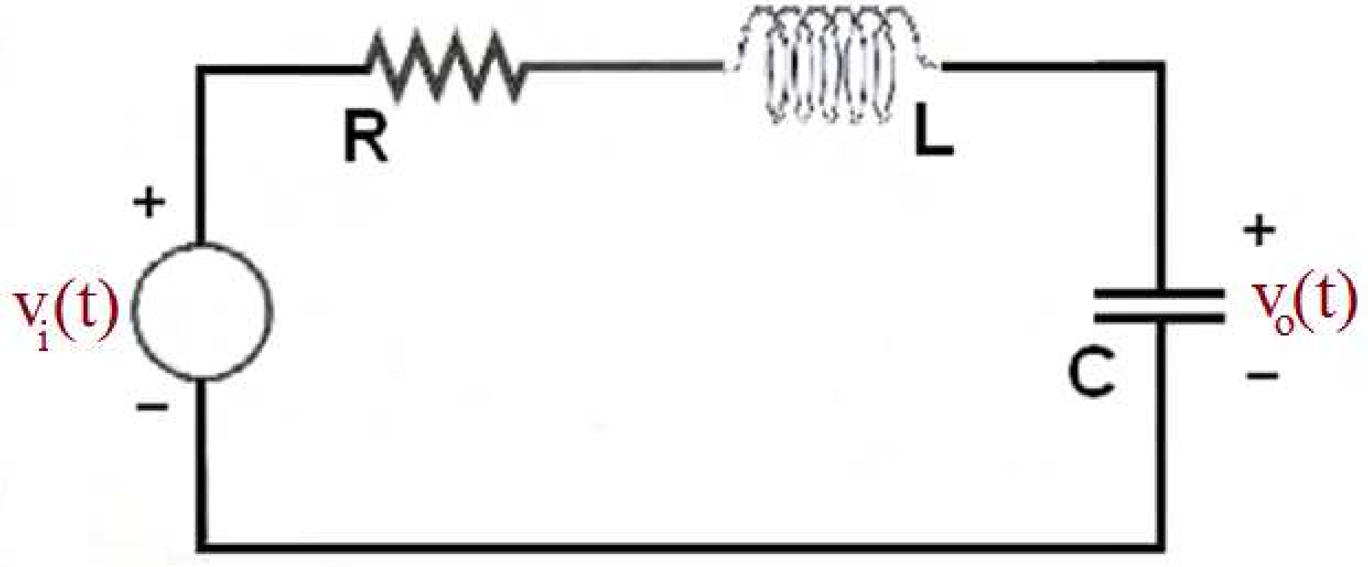


$$\begin{cases} LC\dfrac{d^2v_o}{dt^2} + RC\dfrac{dv_o}{dt} + v_o = LCv_o'' + RCv_o' + v_o = v_i, \\ v_o'(0) = 0, \qquad v_o(0) = 0 \end{cases}$$

hence,

$$LC\,s^2V_o(s) + RC\,s\,V_o(s) + V_o(s) \;=\; V_i(s),$$

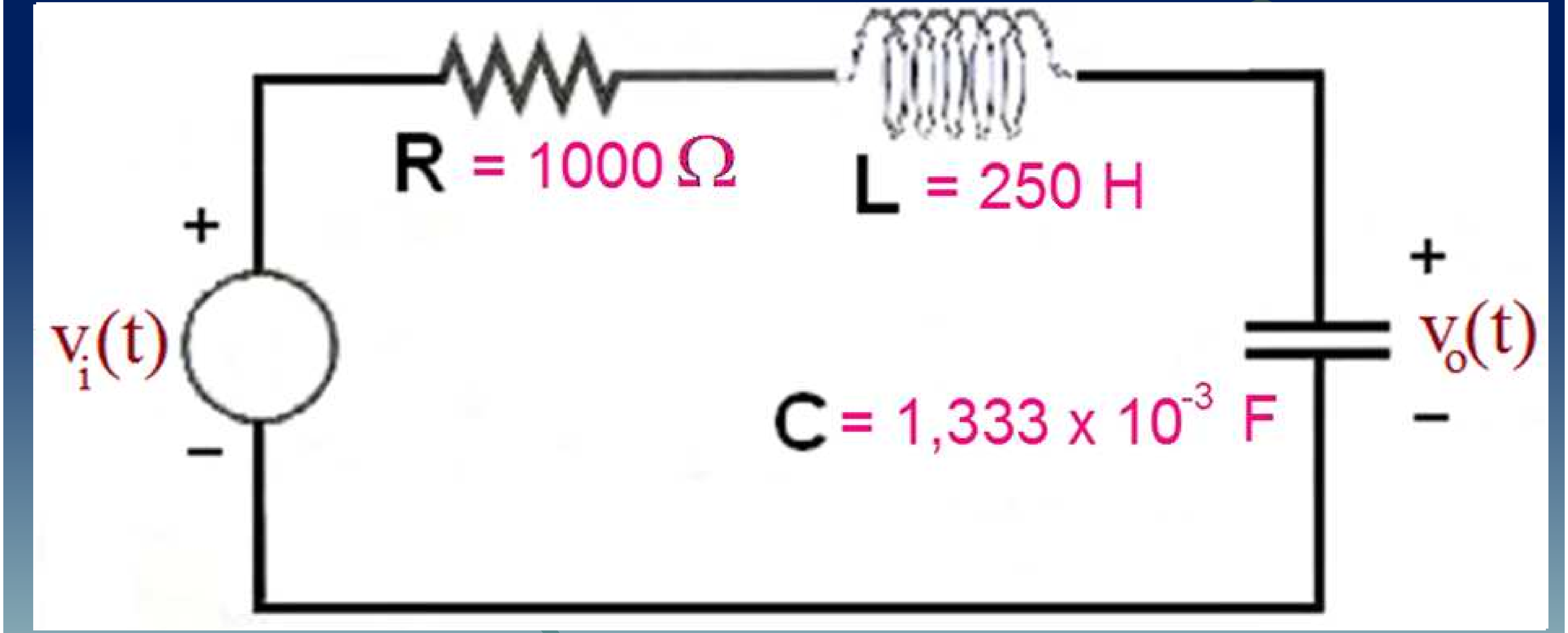
RLC circuit series



thus, the Transfer Function (T.F.) of the system is given by

$$F.T. = \frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

RLC circuit series

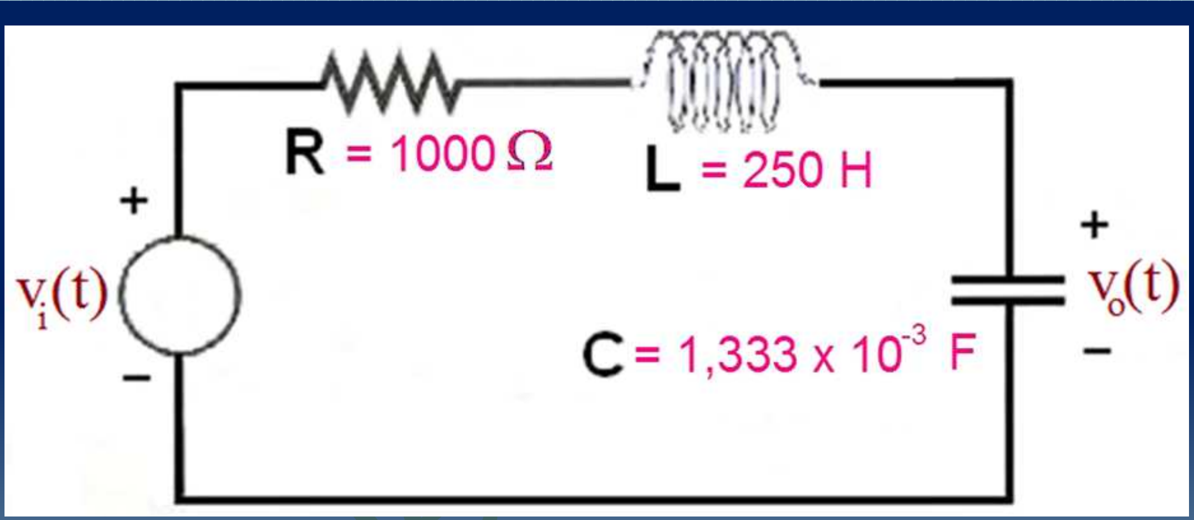


R = 1000 Ω

L = 250 H

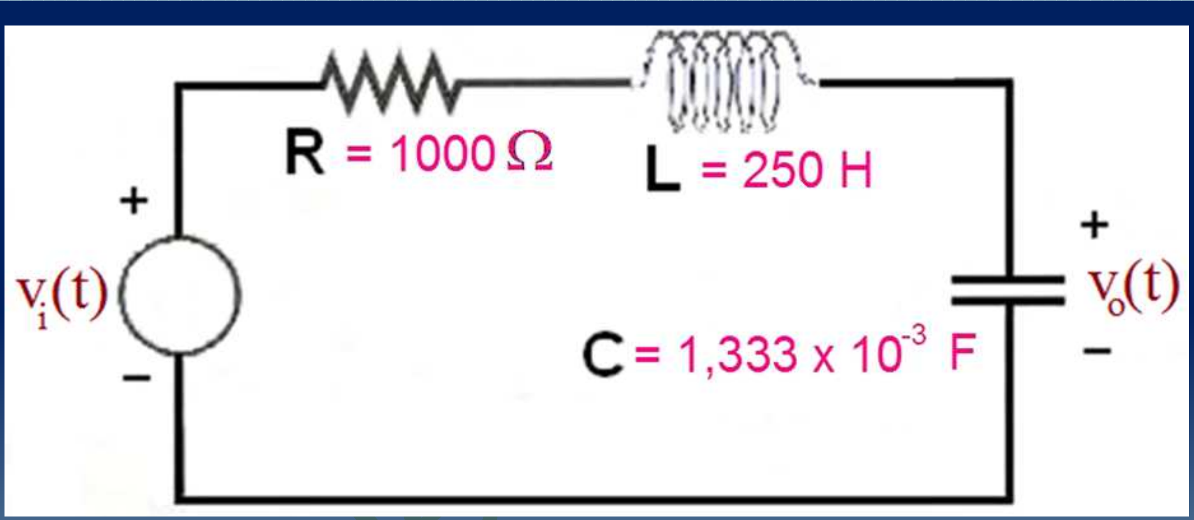
C = 1,333 x 10^{-3} F

RLC circuit series



$$\begin{cases} \dfrac{d^2 v_o}{dt^2} + 4\dfrac{dv_o}{dt} + 3v_o = v_o'' + 4v_o' + 3\,v_o = 3v_i, \\ v_o'(0) = 0, \qquad v_o(0) = 0 \end{cases}$$

RLC circuit series

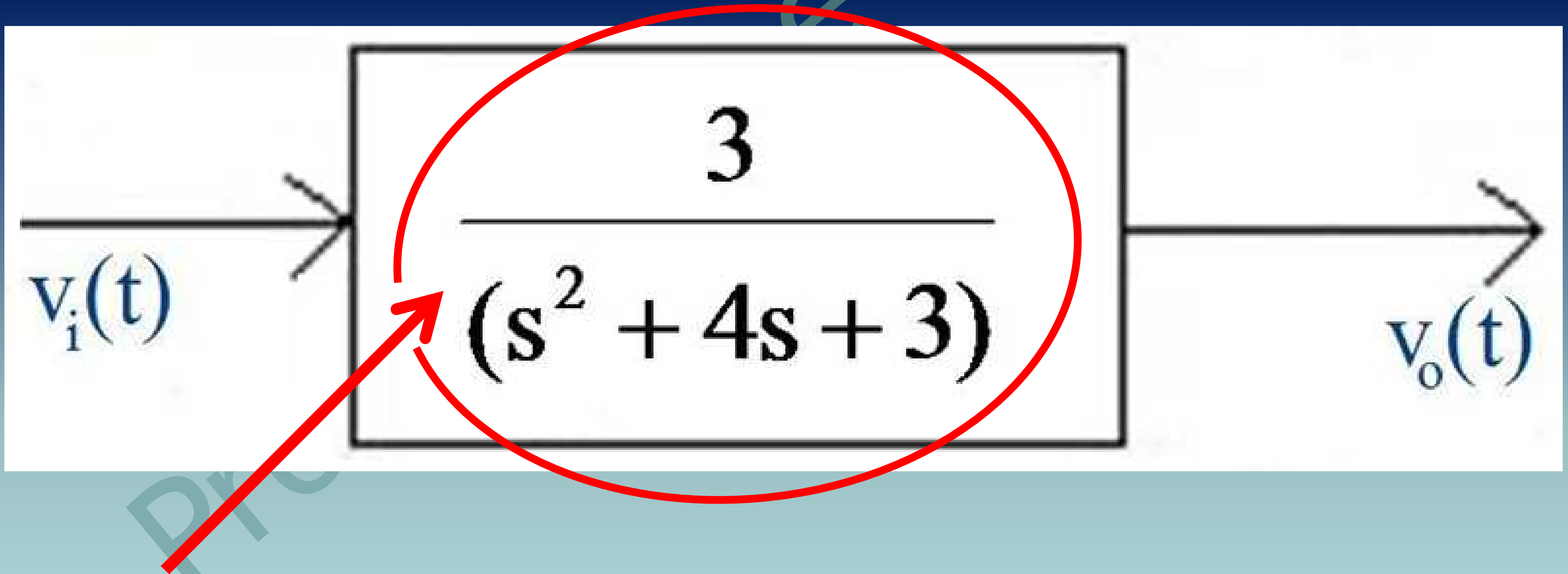


and therefore the Transfer Function (T.F.) of the system will be:

$$\text{F.T.} = \frac{V_o(s)}{V_i(s)} = \frac{3}{s^2 + 4s + 3}$$

Systems Representation

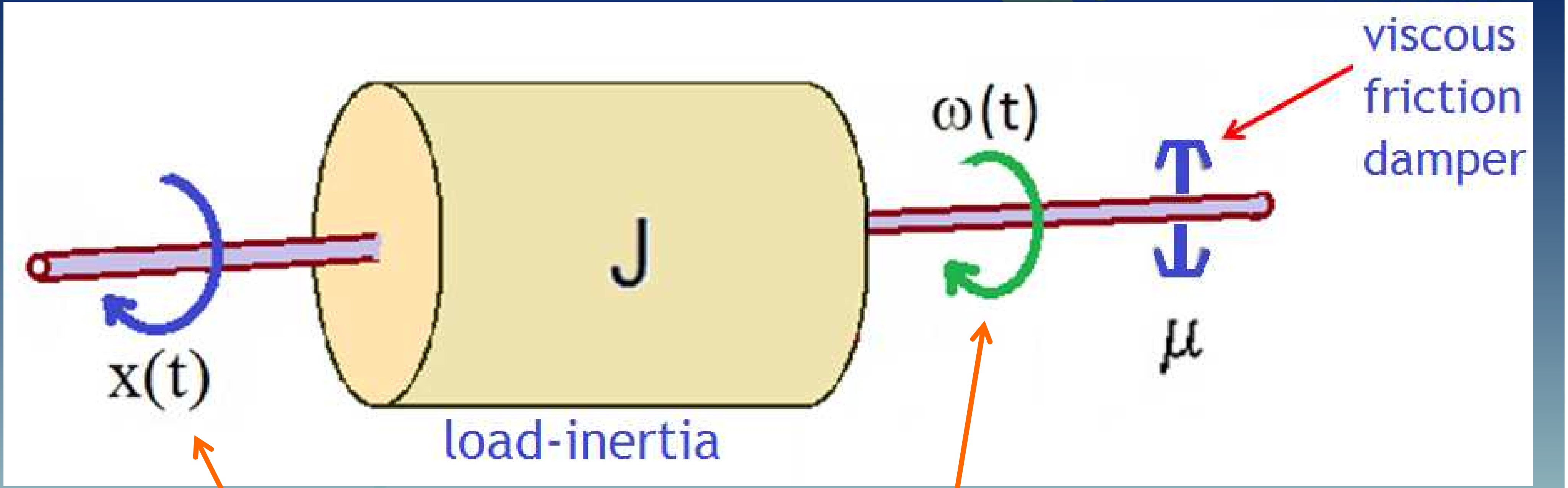
RLC circuit series



Transfer Function (T.F.)
of the system

rotational mechanical system

rotational mechanical system

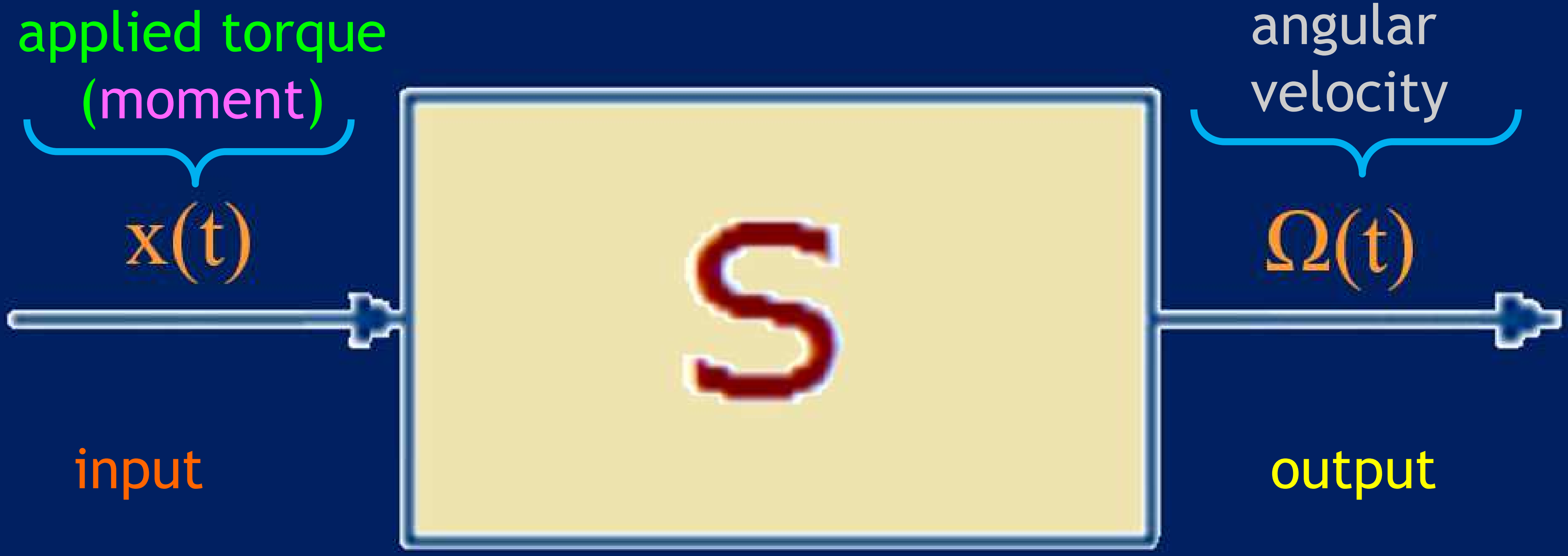


applied
torque
(moment)

angular
velocity

Systems Representation

rotational mechanical system



applied torque

(moment)

$x(t)$

input

S

angular velocity

$\Omega(t)$

output

$$\text{F.T.} = \frac{\Omega(s)}{X(s)}$$

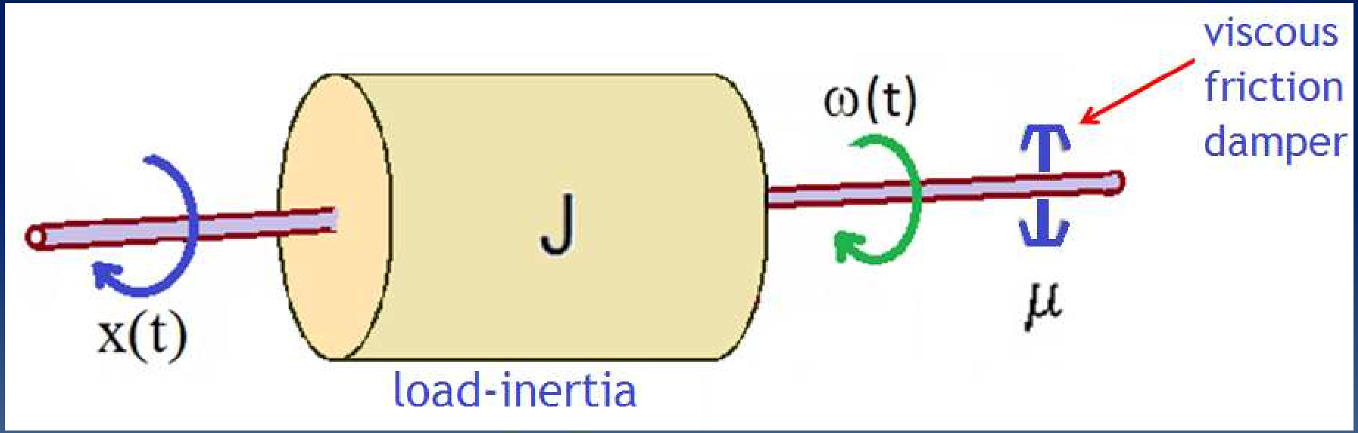
output

input

$\Omega(s)$ = Laplace Transform of $\omega(t)$

$X(s)$ = Laplace Transform of $x(t)$

rotational mechanical system

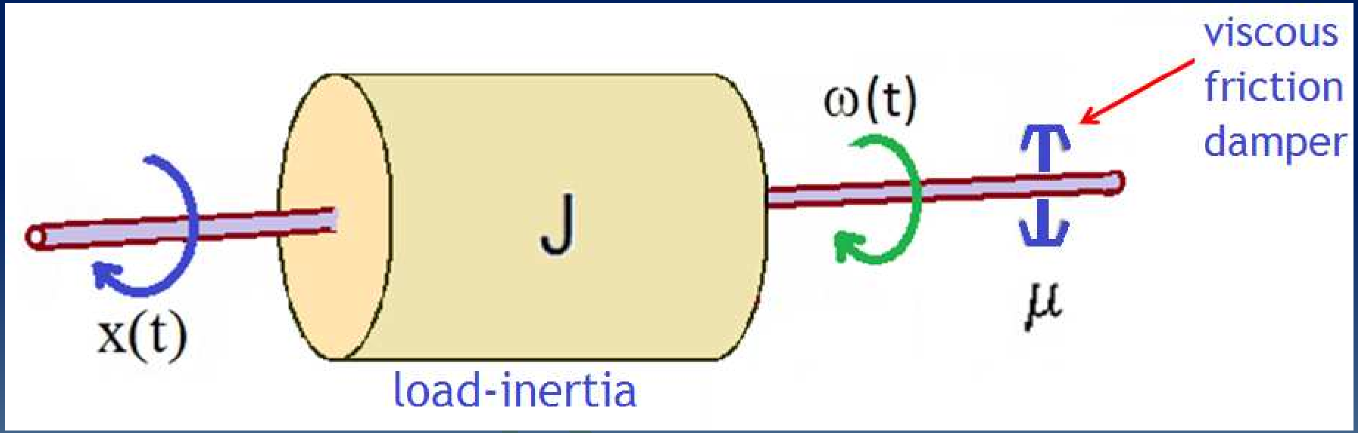


$$J\,\omega' + \mu\,\omega = x\,,$$

or

$$J\,\frac{d\omega}{dt} + \mu\,\omega(t) = x\,,$$

rotational mechanical system

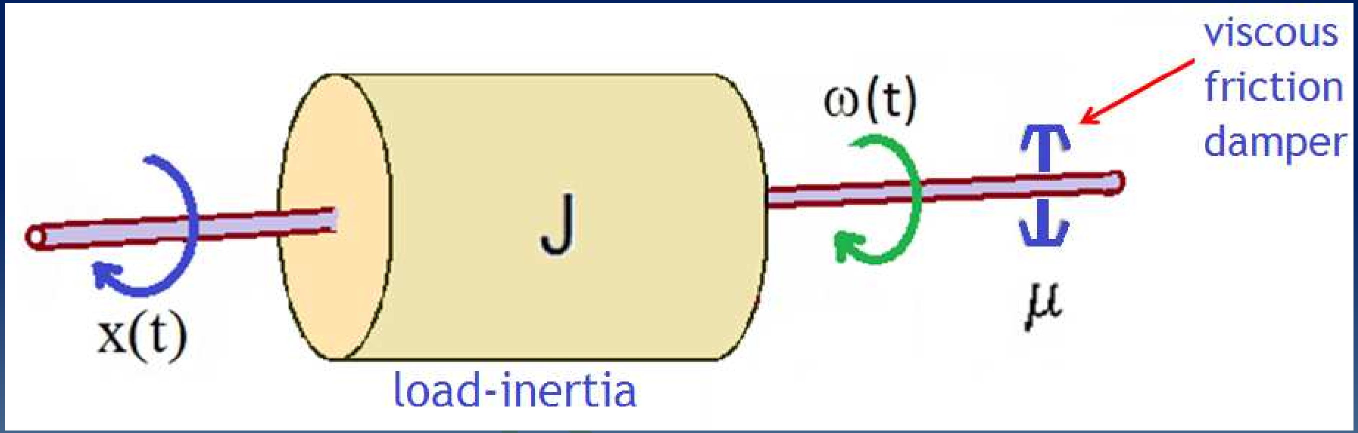


$$\begin{cases} J\dfrac{d\omega}{dt} + \mu\,\omega = J\omega' + \mu\omega = x \\ \omega(0) = 0 \end{cases}$$

hence,

$$J\,s\,\Omega(s) + \mu\,\Omega(s) = X(s),$$

rotational mechanical system



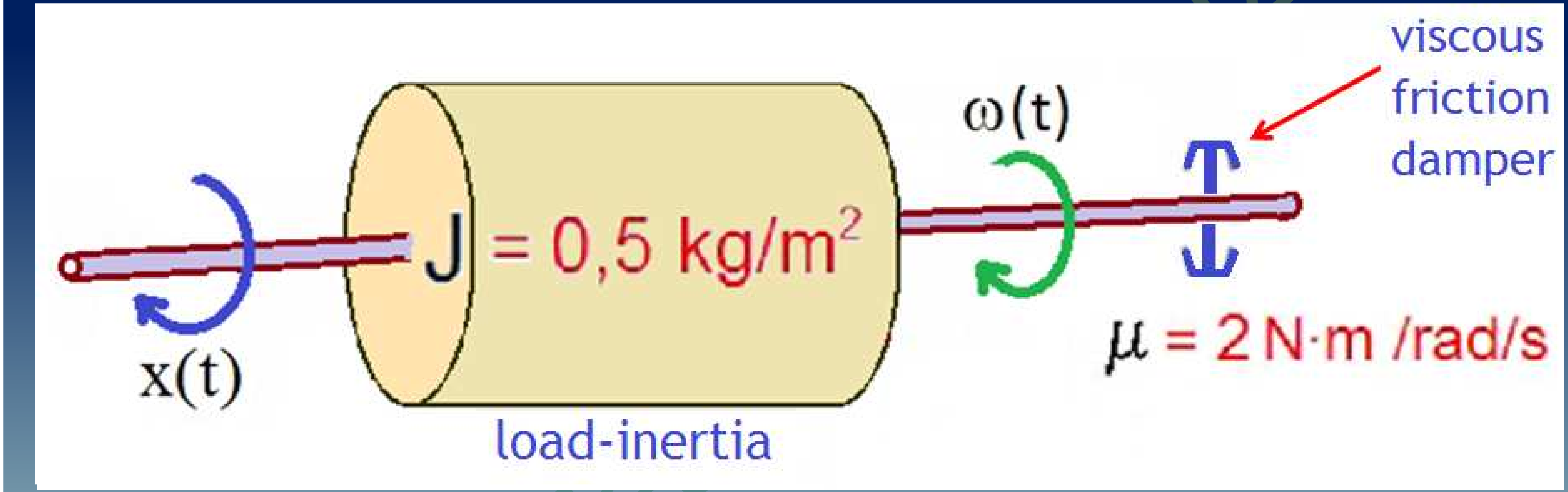
and therefore, the Transfer Function (T.F.) of the system is given by

$$\text{F.T.} = \frac{\Omega(s)}{X(s)} = \frac{1}{Js + \mu}$$

rotational mechanical system

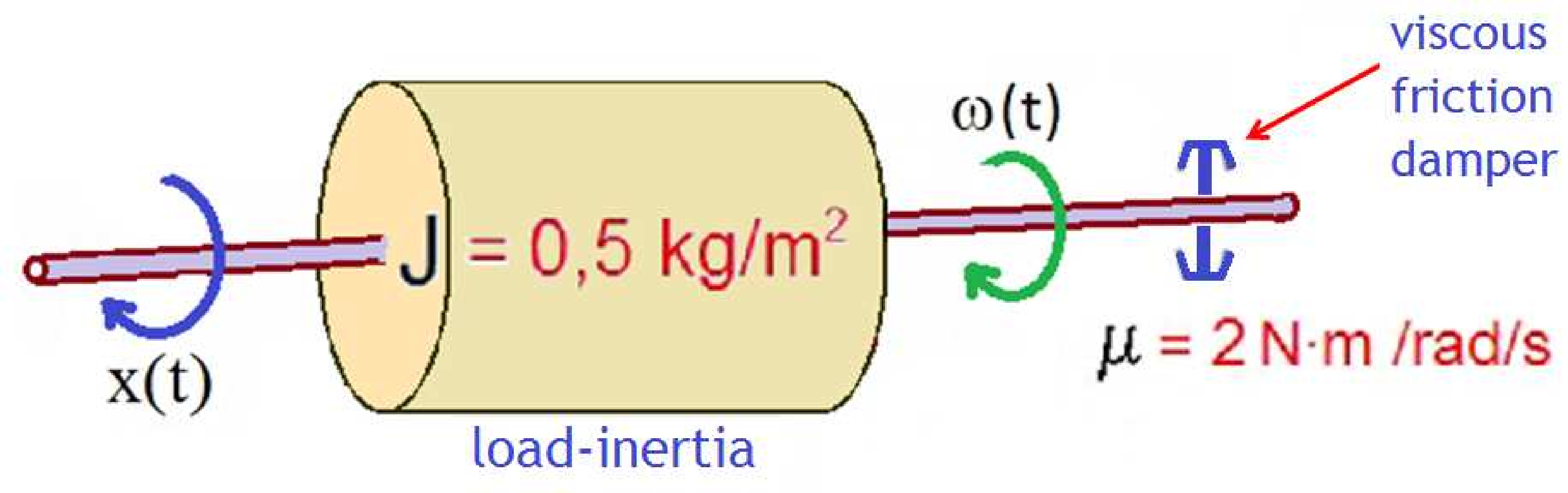
$J = 0{,}5\ \mathrm{kg/m^2}$

$\mu = 2\ \mathrm{N{\cdot}m\ /rad/s}$



$$\begin{cases} \dfrac{d\omega}{dt} + 4\,\omega = \omega' + 4\,\omega = 2\,x\,, \\ \omega(0) = a \end{cases}$$

rotational mechanical system

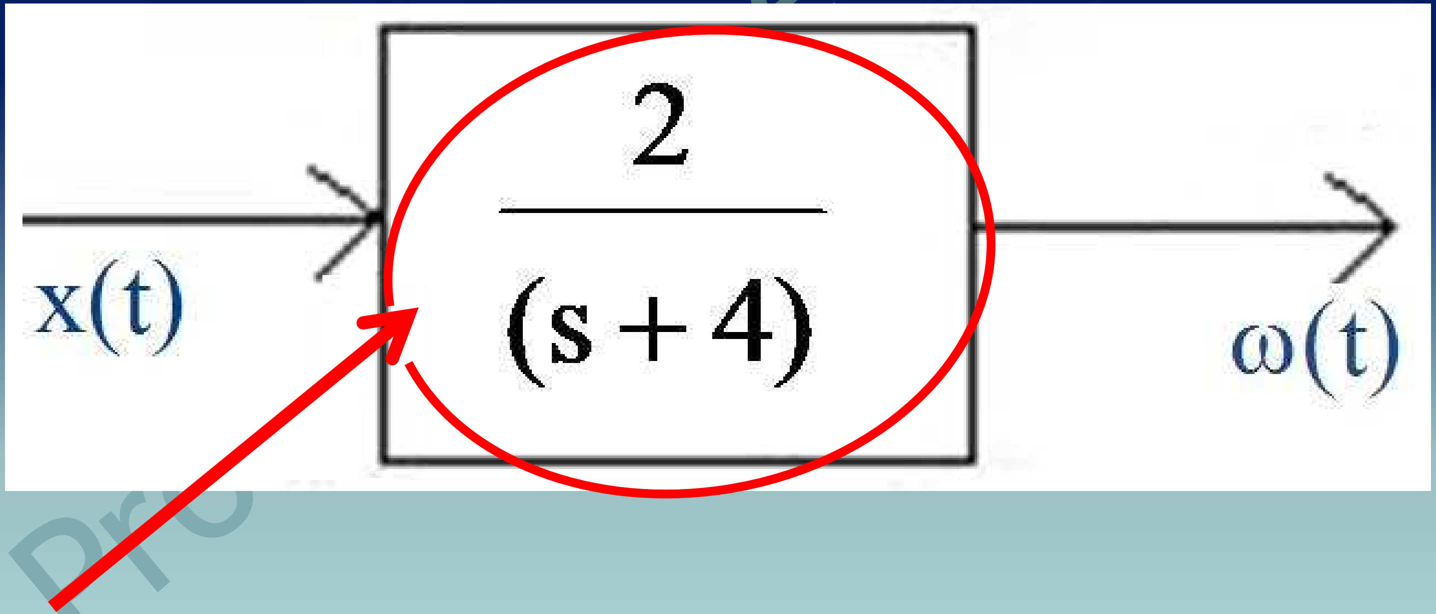


thus, Transfer Function (T.F.) of the system is

$$\text{T.F.} = \frac{\Omega(s)}{X(s)} = \frac{2}{s+4}$$

Systems Representation

rotational mechanical system

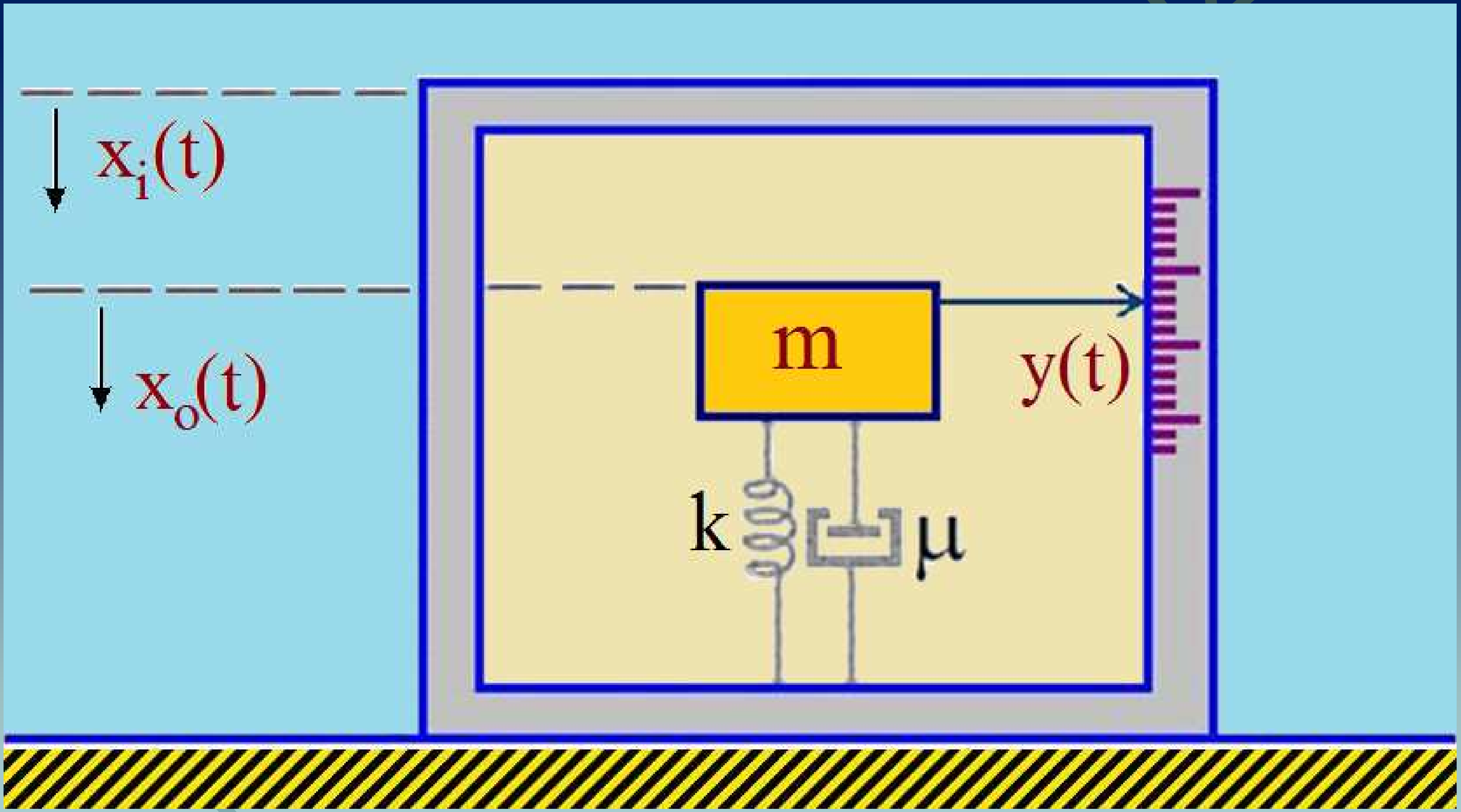


$$\frac{2}{(s+4)}$$

Transfer Function (T.F.)
of the system

seismograph

seismograph



Systems Representation

seismograph

displacement of the box

$x_i(t)$

input

S

displacement of the m mass

$y(t)$

output

$$\boxed{\text{F.T.} = \frac{Y(s)}{X_i(s)}}$$

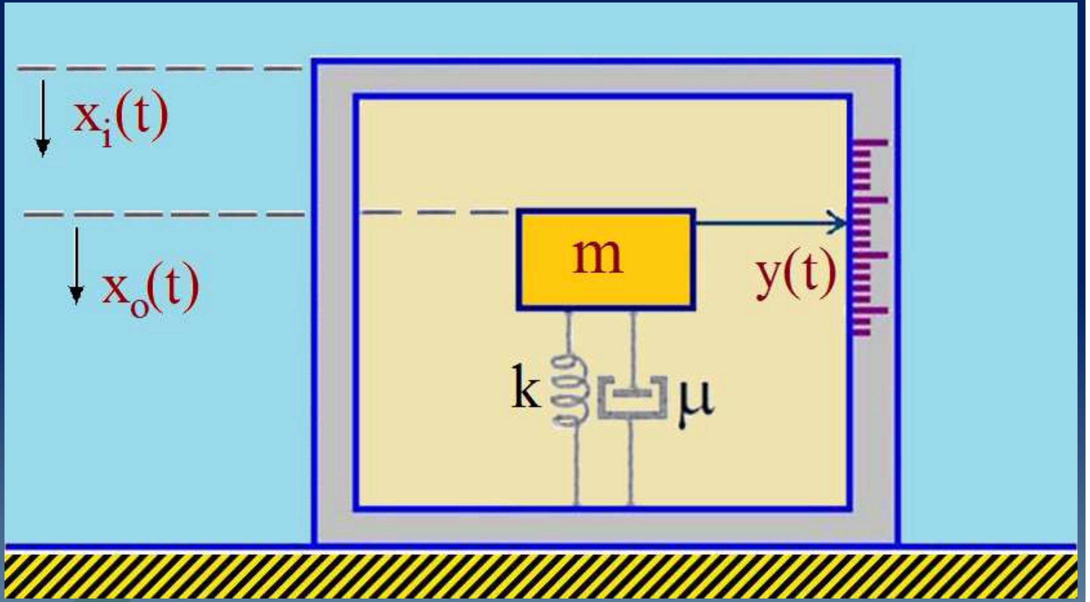
output

input

$X_i(s)$ = Laplace Transform of $x_i(t)$

$Y(s)$ = Laplace Transform of $y(t)$

seismograph

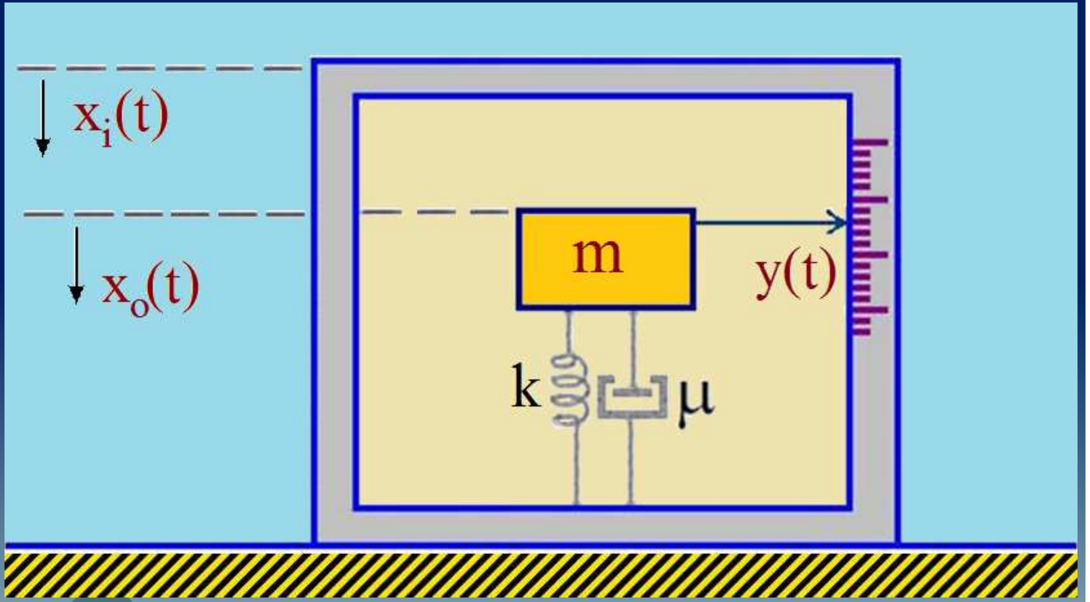


$$m\,y'' + \mu\,y' + k\,y = -m\,x_i'',$$

or

$$m\frac{d^2y}{dt^2} + \mu\frac{dy}{dt} + k\,y = -m\frac{d^2x_i}{dt^2},$$

seismograph

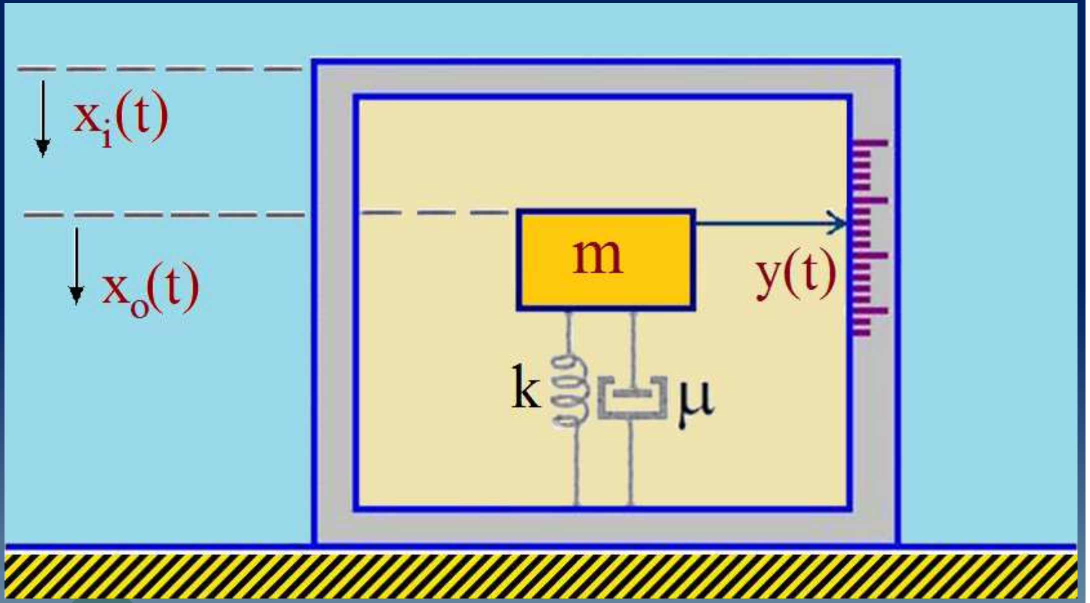


$$\begin{cases} m\dfrac{d^2y}{dt^2} + \mu\dfrac{dy}{dt} + ky = m\,y'' + \mu\,y' + k\,y = -m\,x_i'' \,, \\ y(0) = 0\,, \quad y'(0) = 0 \end{cases}$$

hence,

$$m\,s^2 Y(s) + \mu\,s\,Y(s) + k\,Y(s) = -m\,s^2\,X_i(s),$$

seismograph

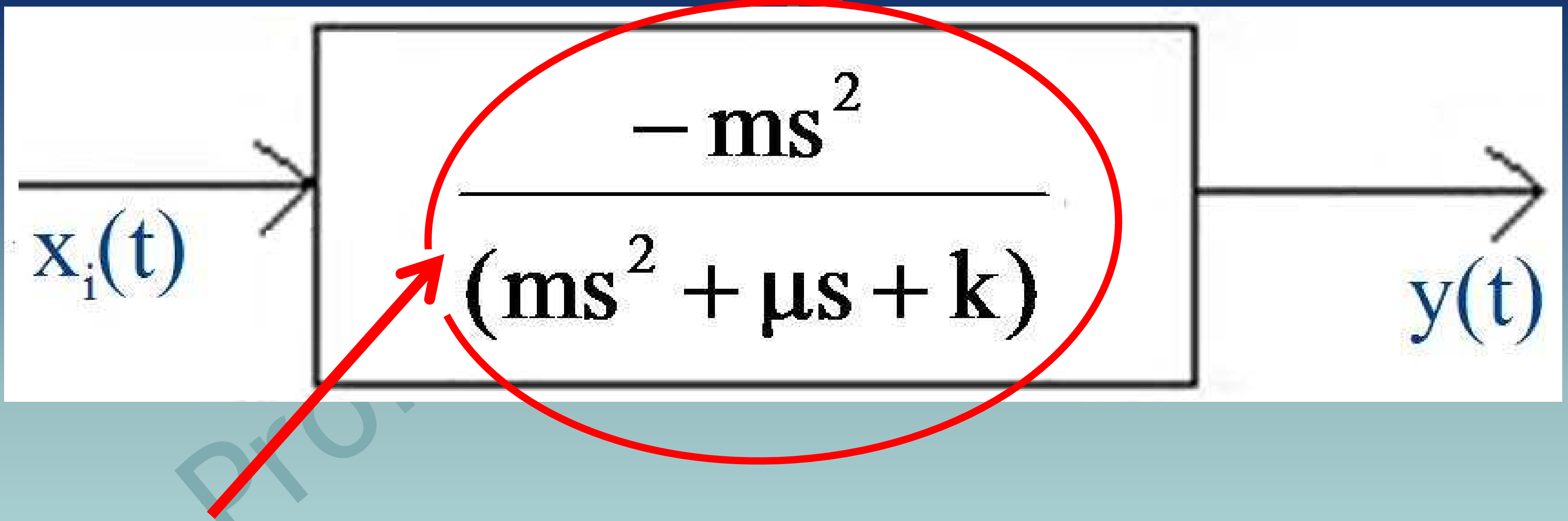


and therefore, the Transfer Function (T.F.) of the system is given by

$$\text{T.F.} = \frac{Y(s)}{X_i(s)} = \frac{-ms^2}{ms^2 + \mu s + k}$$

Systems Representation

seismograph



$$\frac{-ms^2}{(ms^2+\mu s+k)}$$

Transfer Function (T.F.)
of the system

Transfer Function (T.F.)

Systems Representation

Observe that the Transfer Function (T.F.) should be expressed as polynomial/polynomial in its final form, that is

$$^{q(s)}/_{p(s)}.$$

$$\text{T.F.} = G(s) = \frac{Y(s)}{R(s)} = \frac{q(s)}{p(s)}$$

Systems Representation

$$\text{T.F.} = G(s) = \frac{q(s)}{p(s)}$$

The *roots* of $q(s)$ are called the zeros of the system.

The *roots* of $p(s)$ are called the poles of the system.

Systems Representation

$$\text{T.F.} = G(s) = \frac{q(s)}{p(s)}$$

The *polynomial* $p(s)$ is called the characteristic polynomial of the system.

The *equation*

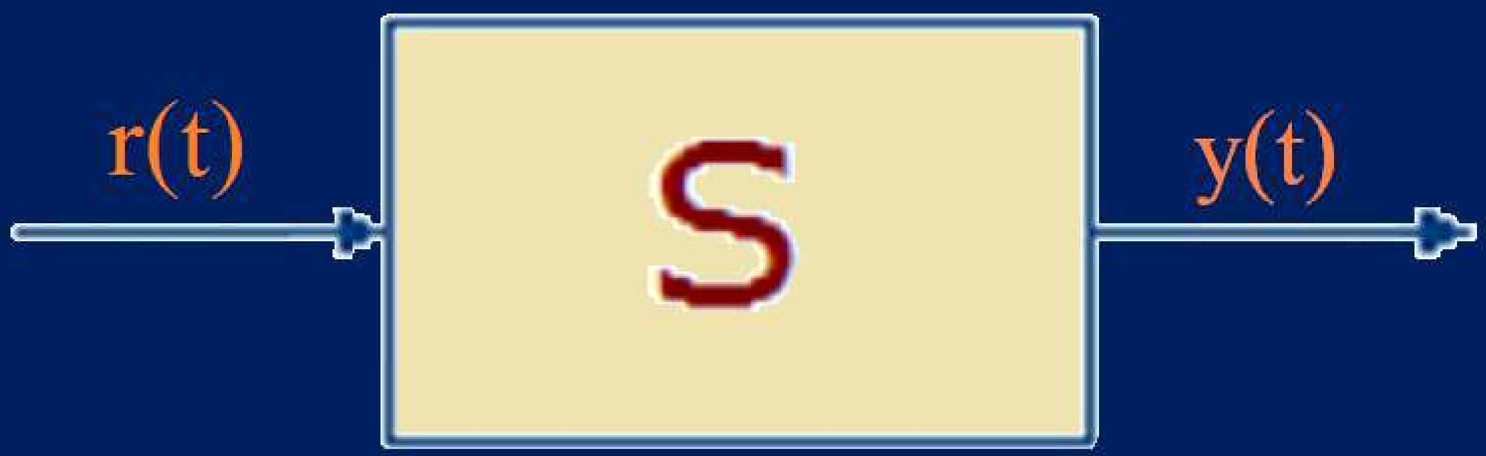
$$p(s) = 0$$

is known as the

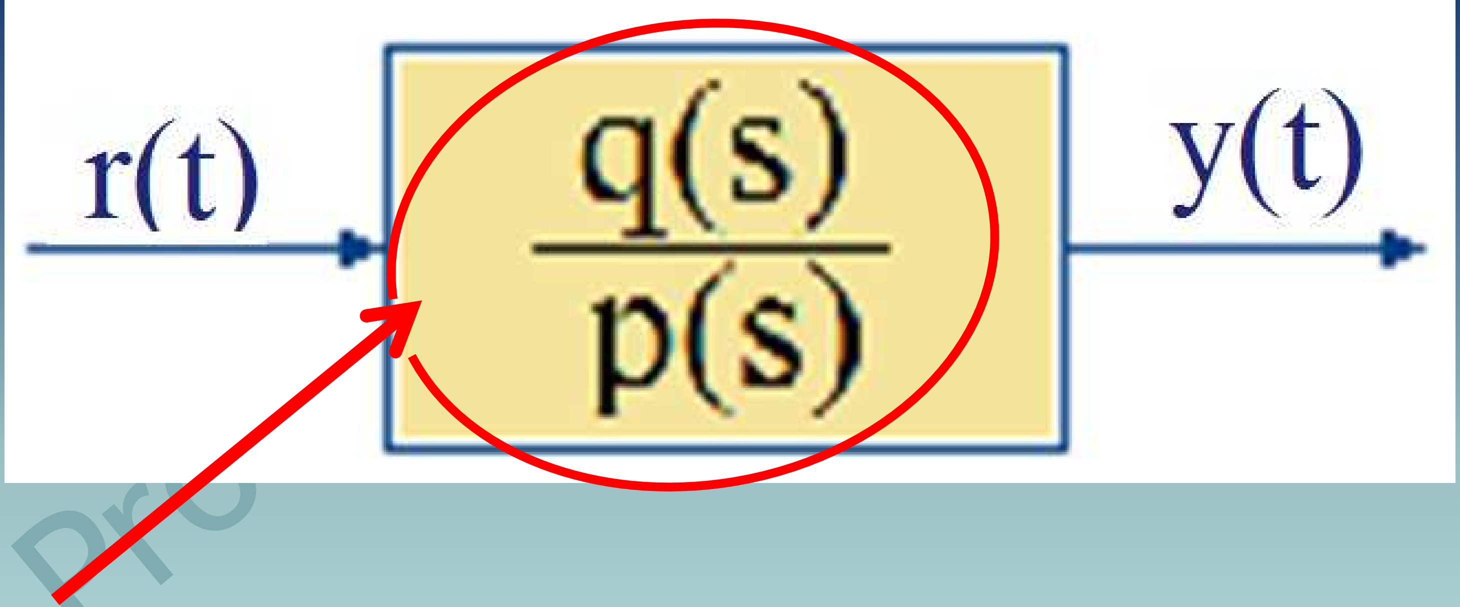
characteristic equation of the system.

Systems Representation

input



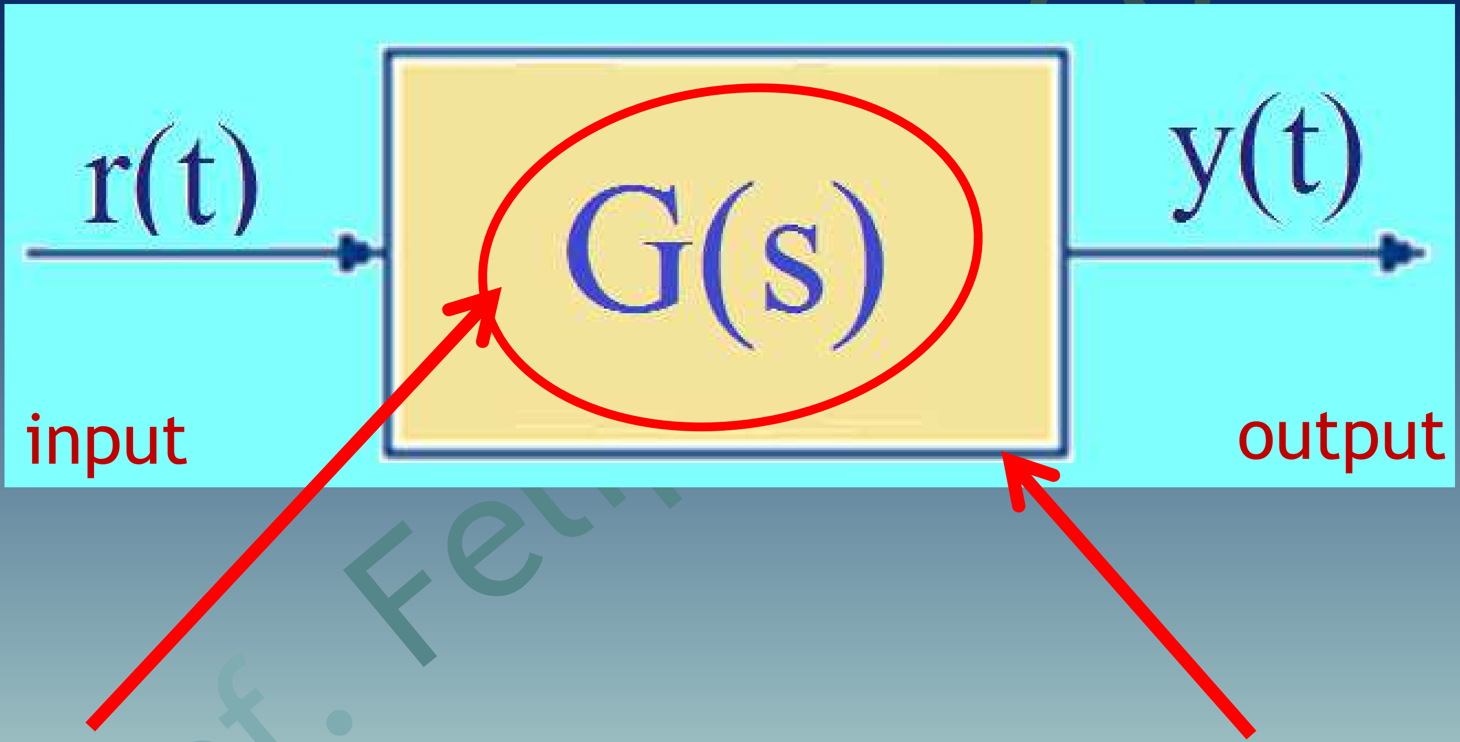
output



Transfer Function (T.F.)
of the system

Systems Representation

or simply,



input

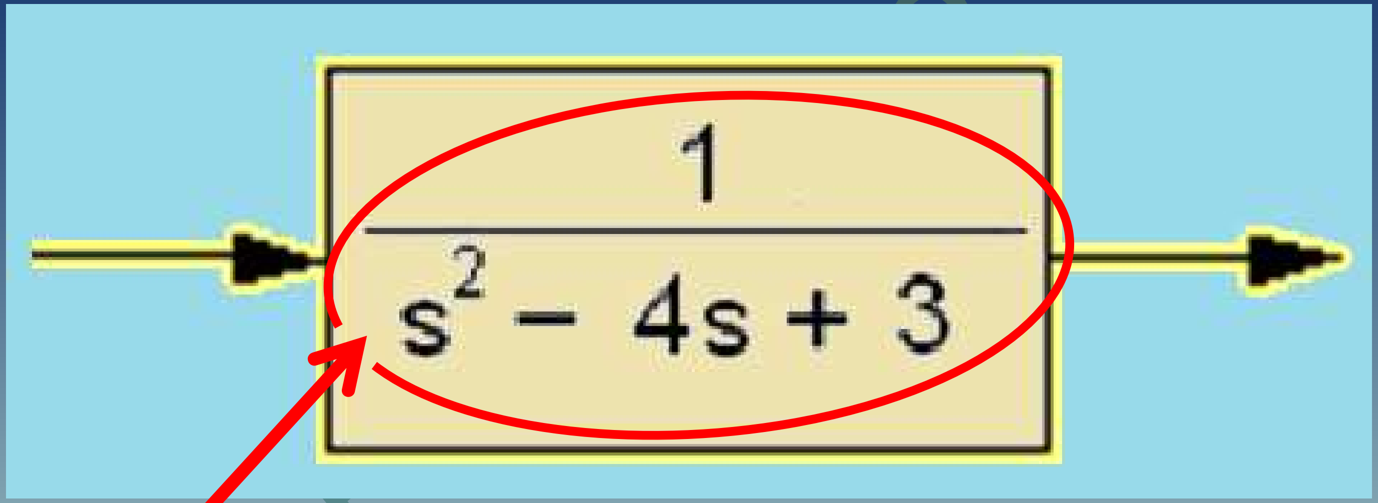
output

Transfer Function (T.F.)
of the system

Single Block or
Black box

Block Diagrams

Having the T.F. we can represent systems with Block Diagrams:


$$\frac{1}{s^2 - 4s + 3}$$

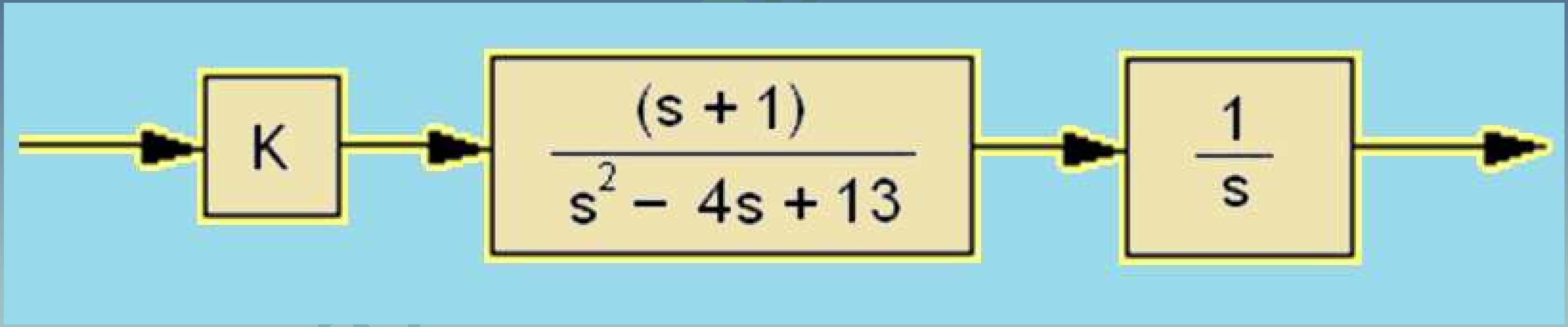
Single Block or black box

Transfer Function (T.F.) of the system

Systems Representation

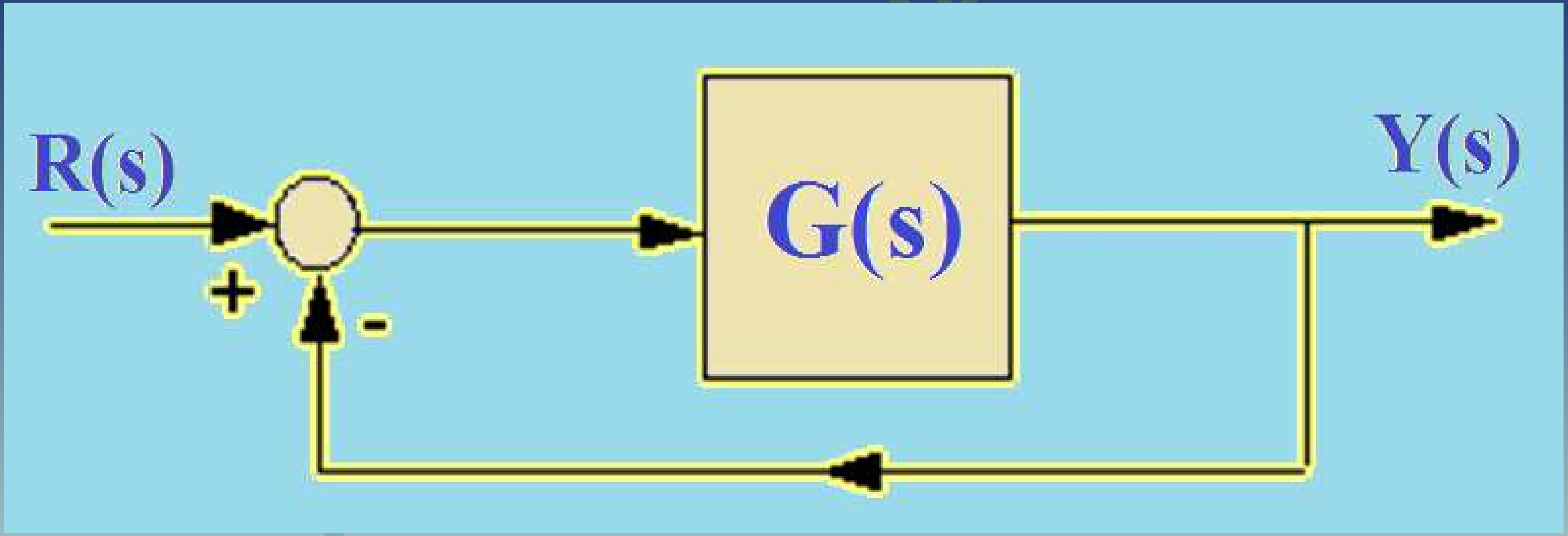
Block Diagrams is the theme of the next chapter.

There are several types of connections with blocks, such as for example, ‘blocks in cascade’:



Blocks in cascade

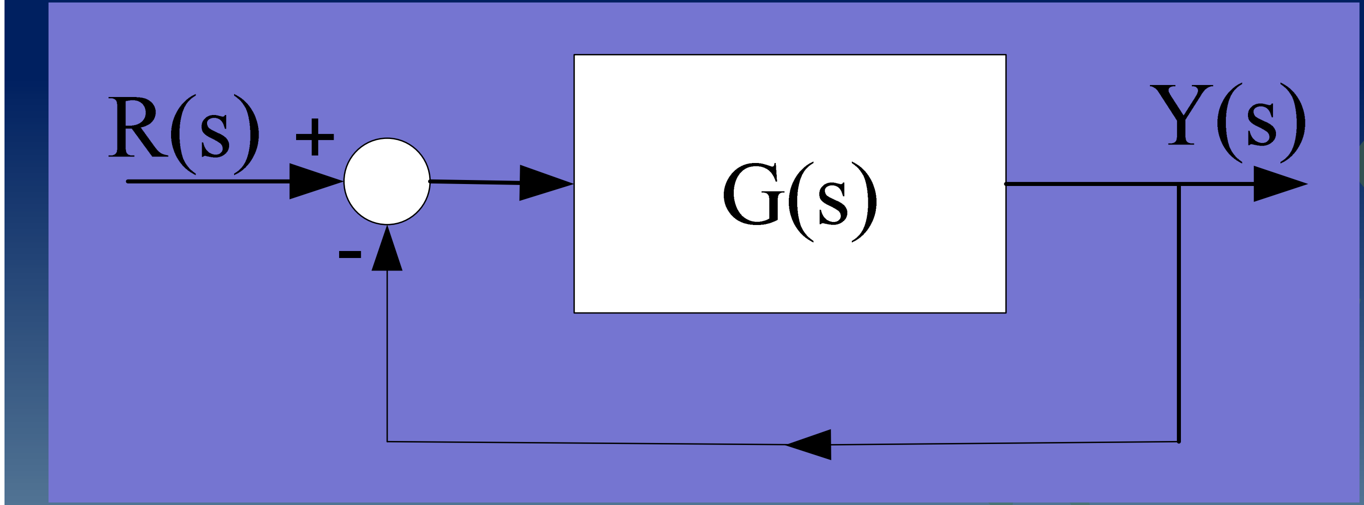
Blocks with feedback:



Block G(s) with unit *feedback*

Example 1:

We'll see in the next chapter that the following block diagram



where:

$$G(s) = \frac{5}{s(s+4)}$$

has the following transfer function:

$$T.F. = \frac{Y(s)}{R(s)} = \frac{\frac{5}{s(s+4)}}{1+\frac{5}{s(s+4)}} = \boxed{\frac{5}{s^2+4s+5}}$$

Example 1 (*continued*):

$$\mathrm{T.F.} = \frac{Y(s)}{R(s)} = \frac{5}{s^2 + 4s + 5}$$

This system has two poles p_1 and p_2 and no zeros.

$$p_1 = -2 + j \qquad p_2 = -2 - j$$

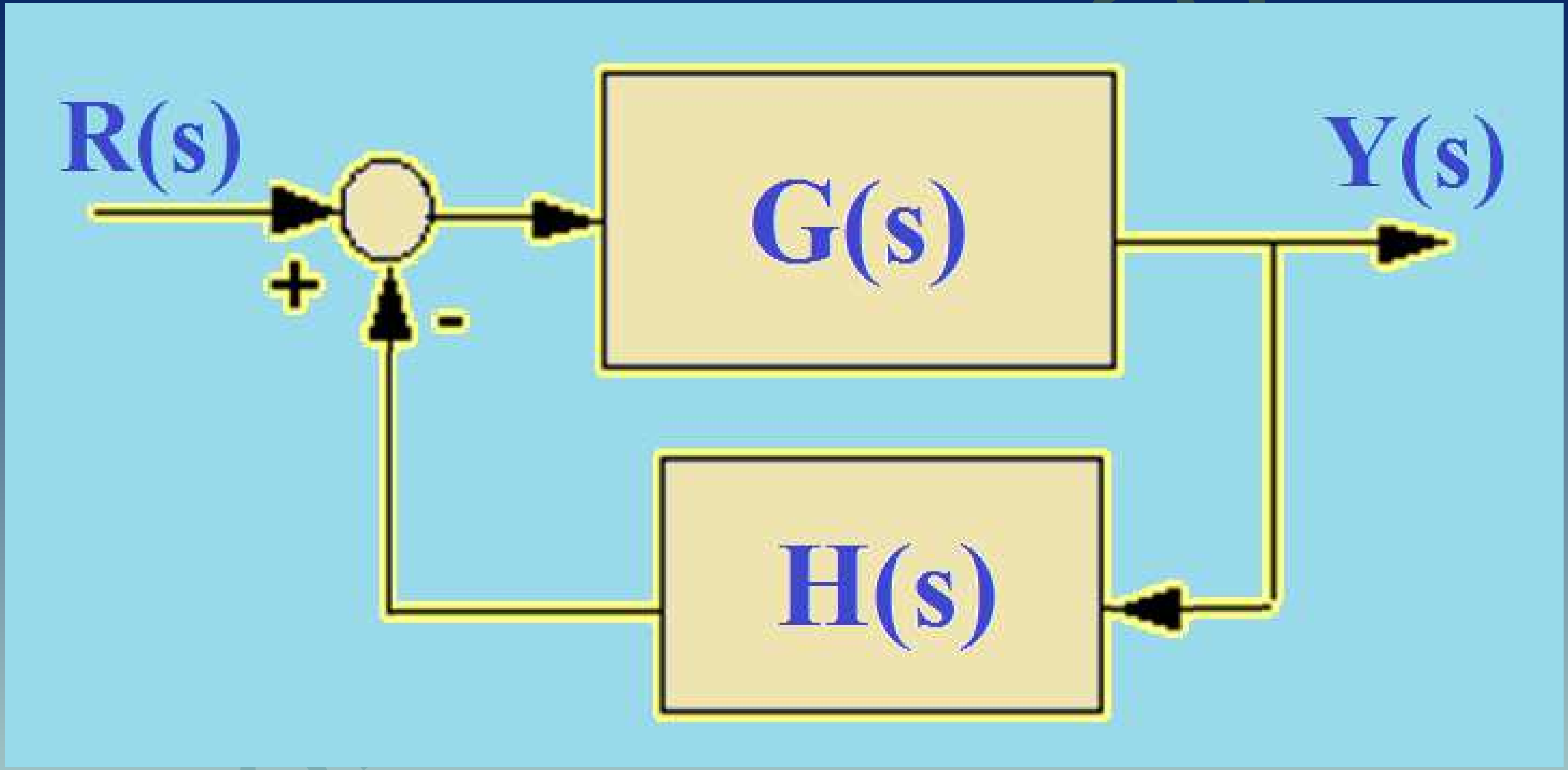
which are the roots of the characteristic polynomial $p(s)$

$$p(s) = s^2 + 4s + 5$$

and the characteristic equation of the system is given by:

$$s^2 + 4s + 5 = 0$$

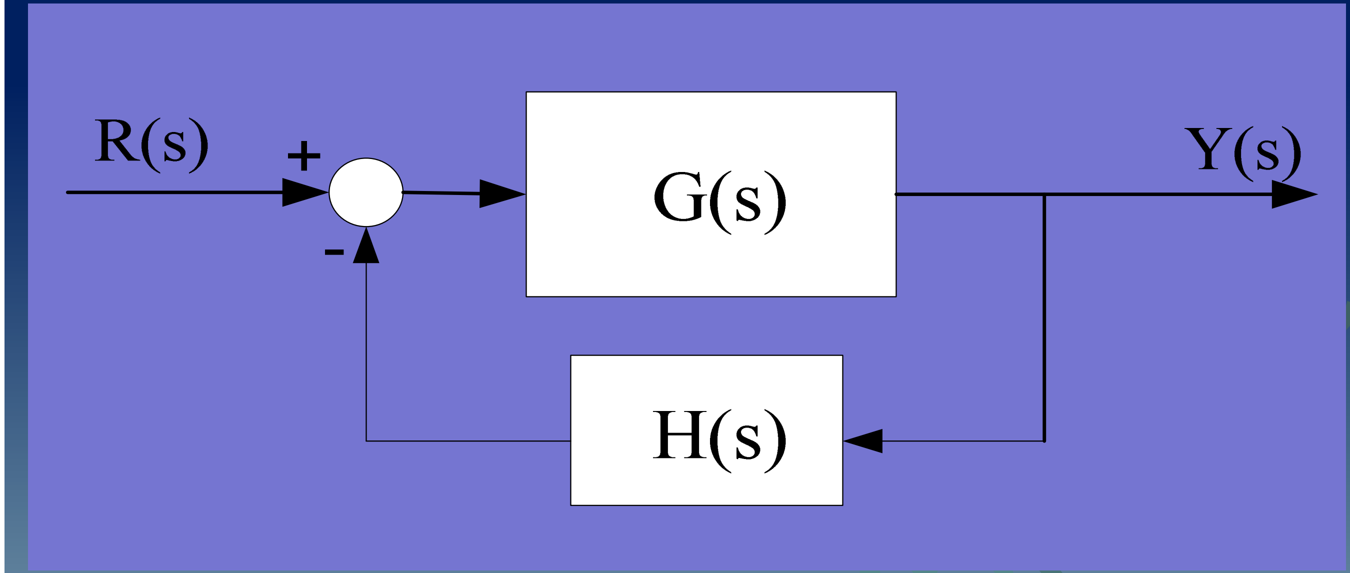
Blocks with feedback:



Block G(s) with <u>non</u> unit *feedback* H(s)

Example 2:

We’ll see in the next chapter that the following block diagram



where:

$$G(s) = \frac{5}{s(s+4)}$$

$$H(s) = \frac{1}{(s+3)}$$

has the following transfer function:

$$T.F. = \frac{Y(s)}{R(s)} = \frac{\frac{5}{s(s+4)}}{1+\frac{5}{s(s+4)}\cdot\frac{1}{(s+3)}} = \boxed{\frac{5(s+3)}{(s^3+7s^2+12s+5)}}$$

Example 2 (*continued*):

$$\text{T. F.} = \frac{Y(s)}{R(s)} = \frac{5(s+3)}{(s^3+7s^2+12s+5)}$$

This system has two poles p_1, p_2 and p_3 and one zero z_1.

$$p_1 = -4{,}65 \qquad p_2 = -1{,}726 \qquad p_3 = -0{,}623 \qquad z_1 = -3$$

which are the roots of the characteristic polynomial $p(s)$

$$p(s) = s^3+7s^2+12s+5$$

and of the equation $s+3=0.$

The characteristic equation of the system is given by:

$$s^3+7s^2+12s+5 = 0$$

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Thank you!

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