Control Systems

3

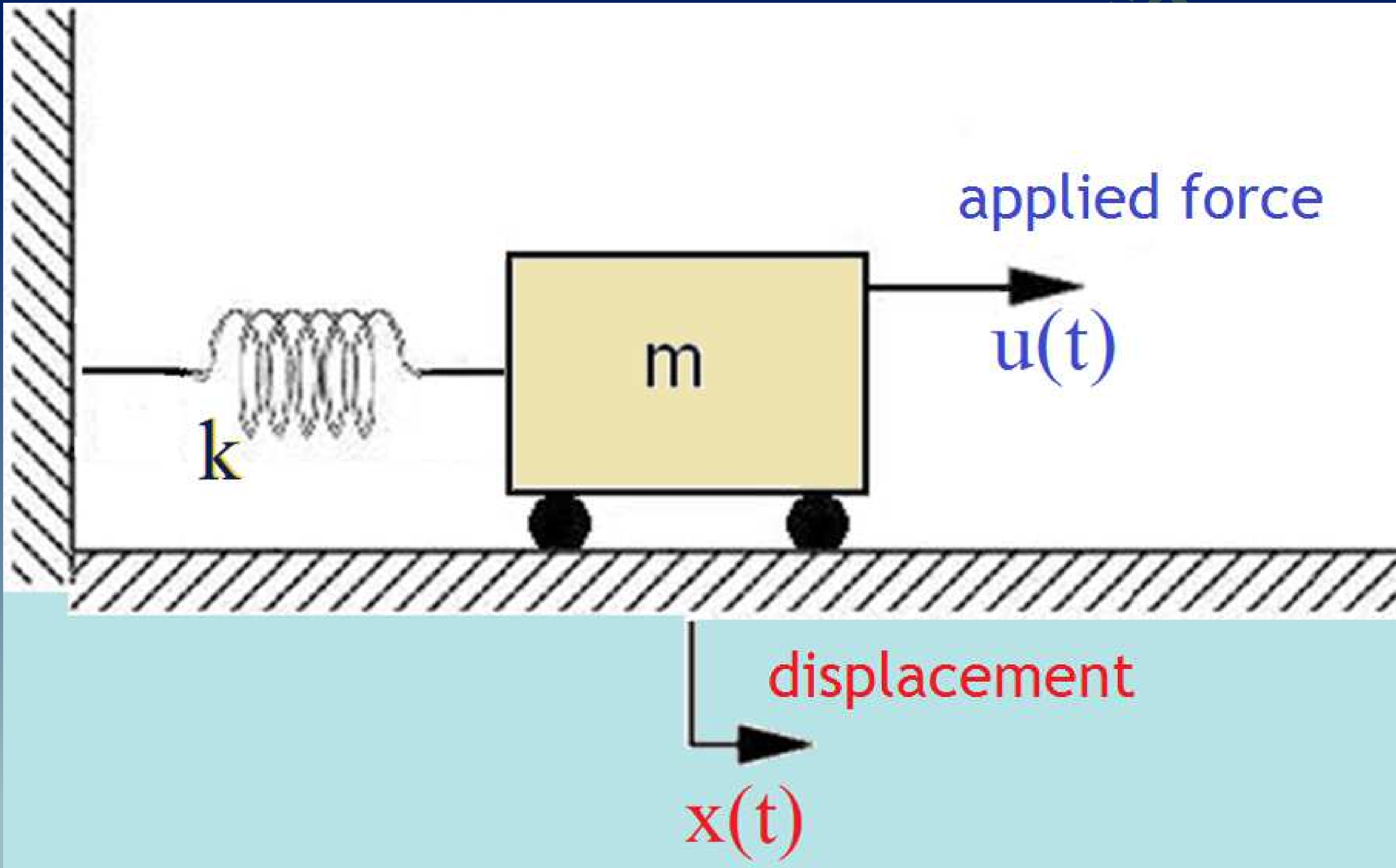
“Systems modelling”

J. A. M. Felippe de Souza

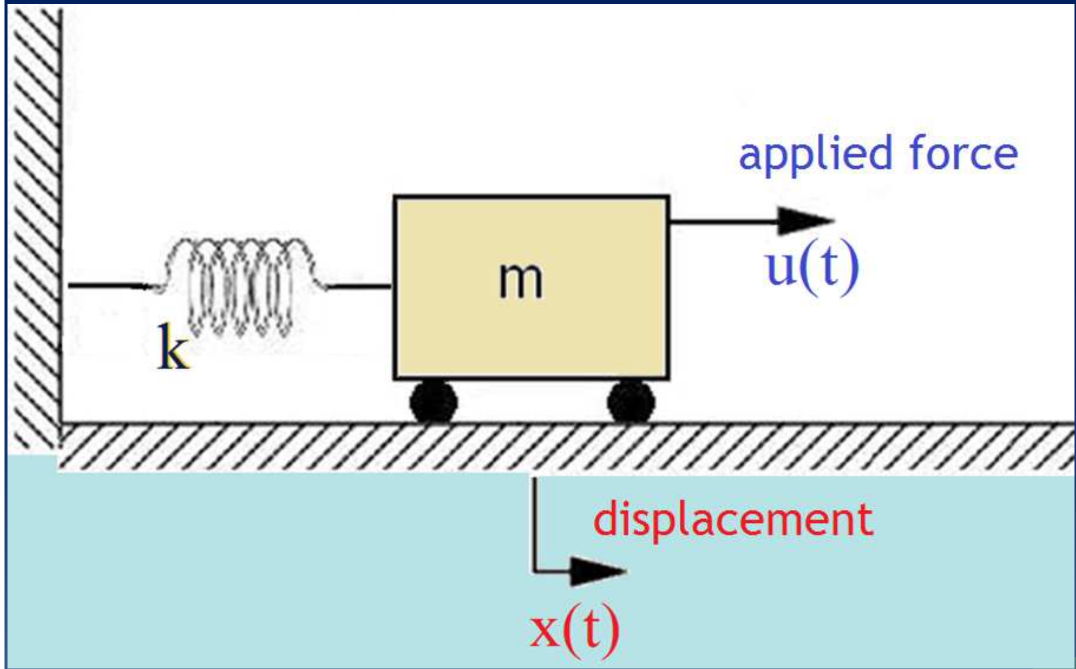

x(t)
S
y(t)
input
output

spring / mass / damper

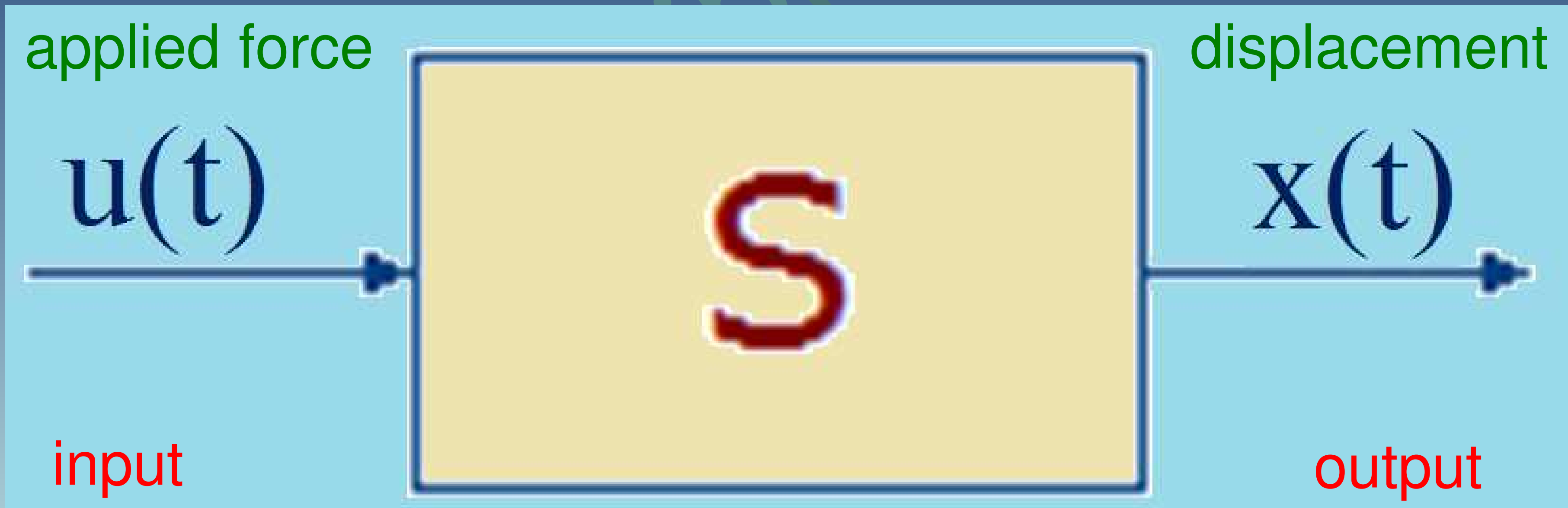
spring / mass / damper



Systems modelling

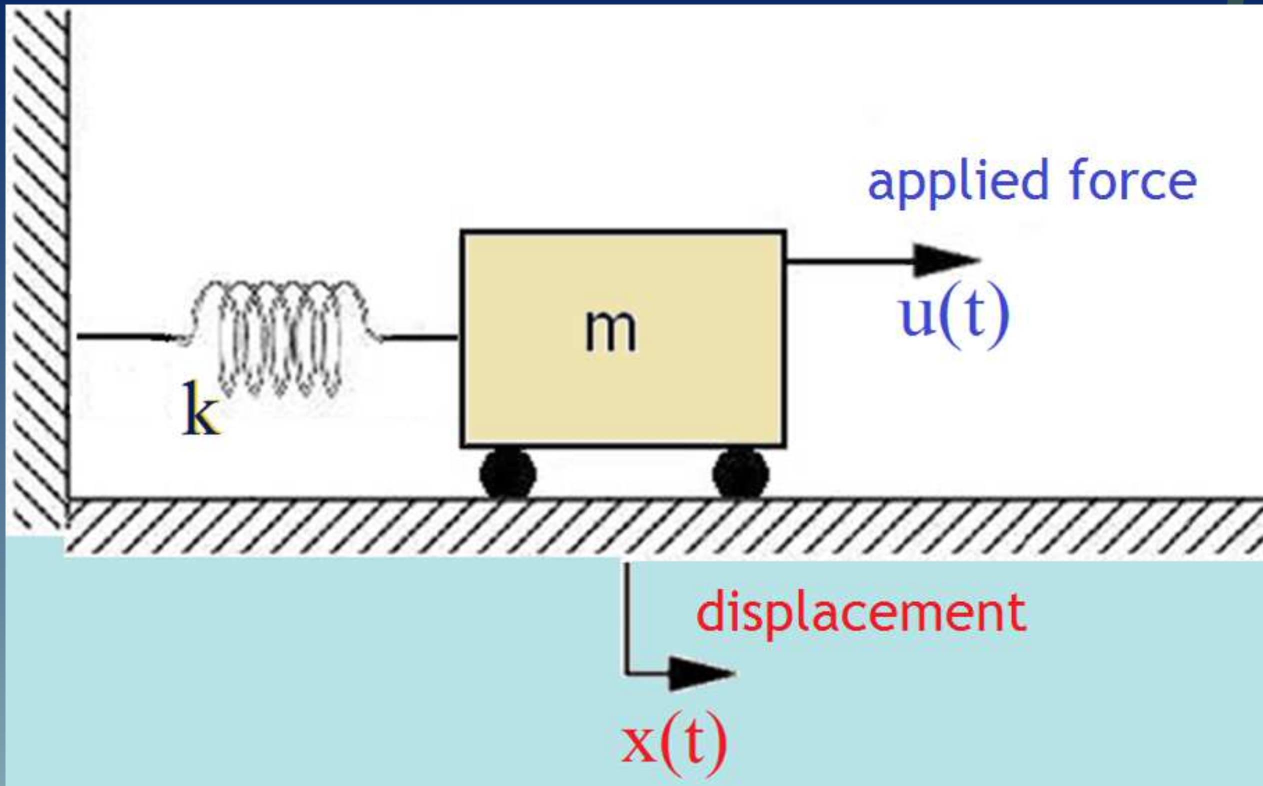


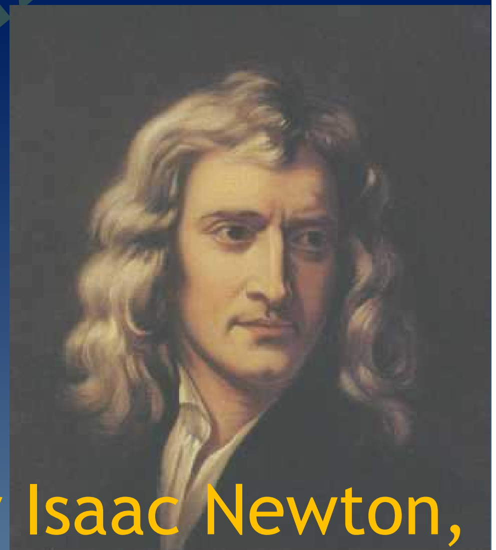
spring / mass / damper



Newton's 2^{nd} Law

$$\sum F = m \cdot a$$



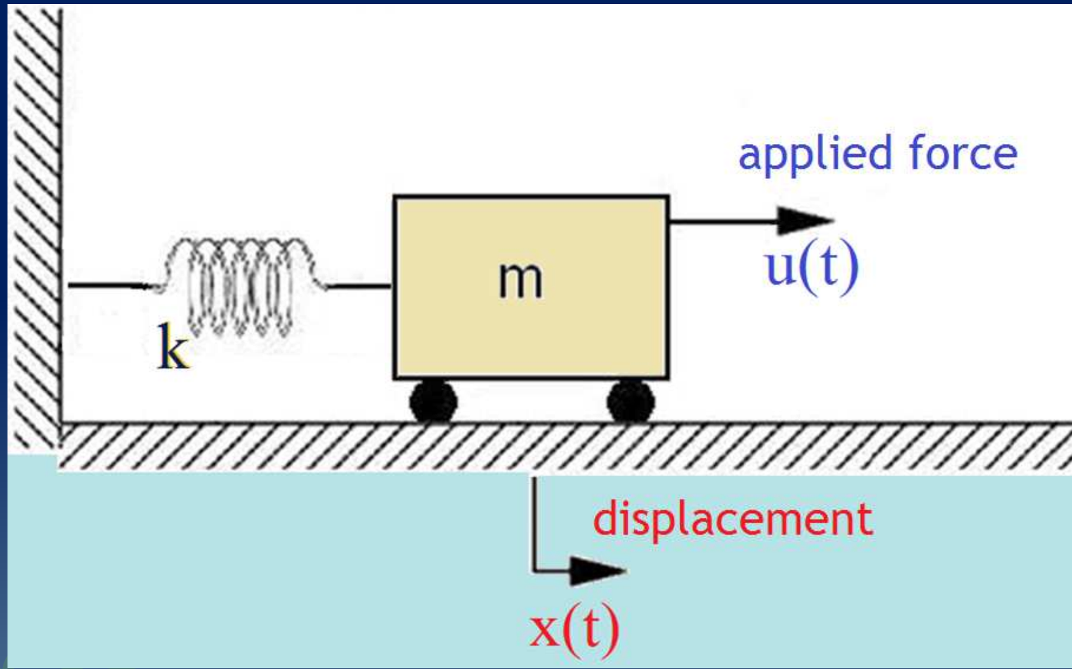


Sir Isaac Newton,
1643-1727

logo

$$m x'' = -kx - \mu x' + u,$$

spring / mass / damper



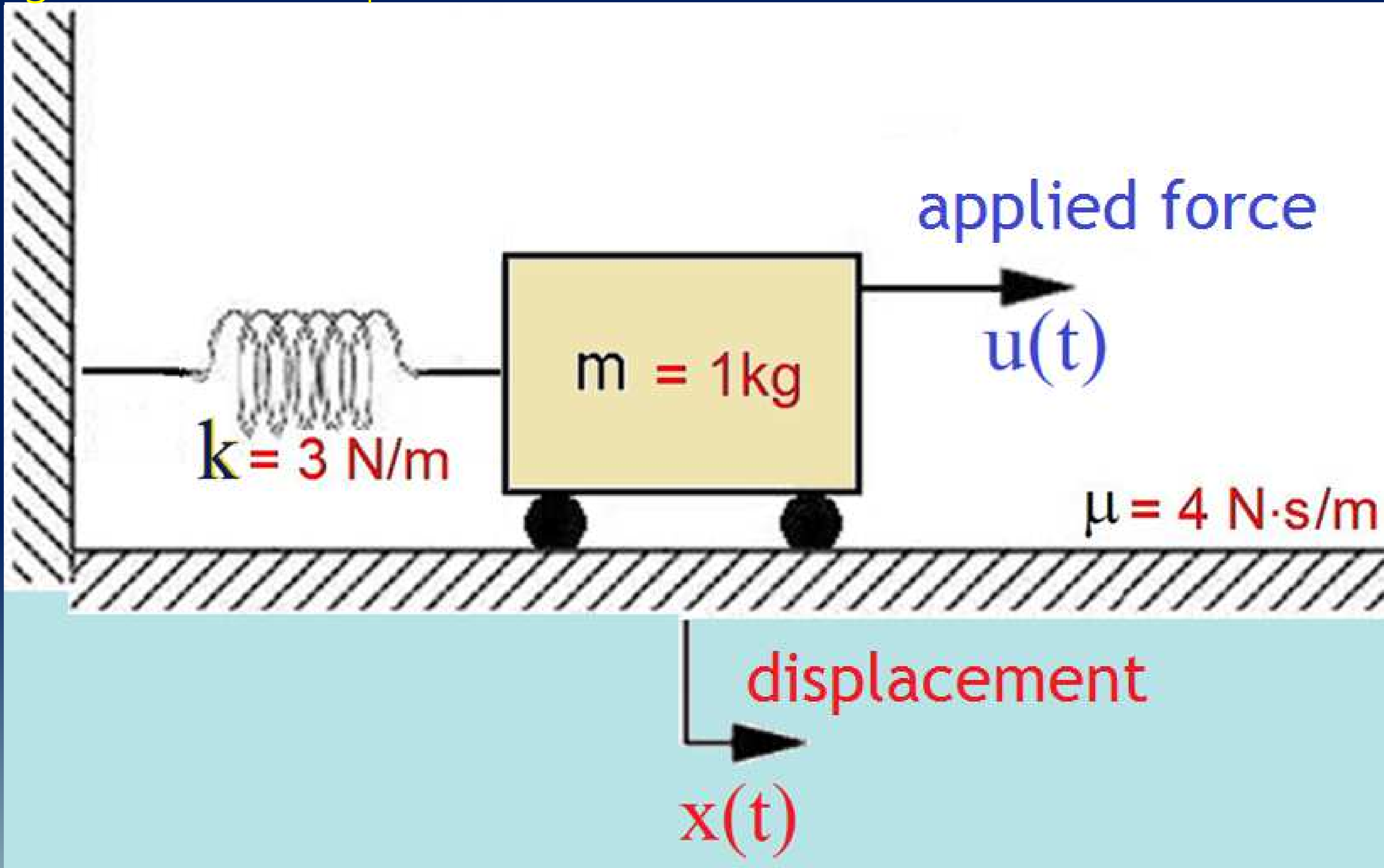
and therefore,

$$m x'' + \mu x' + k x = u,$$

or

$$m \frac{d^2 x}{dt^2} + \mu \frac{dx}{dt} + k\, x = u,$$

spring / mass / damper



m = 1 kg

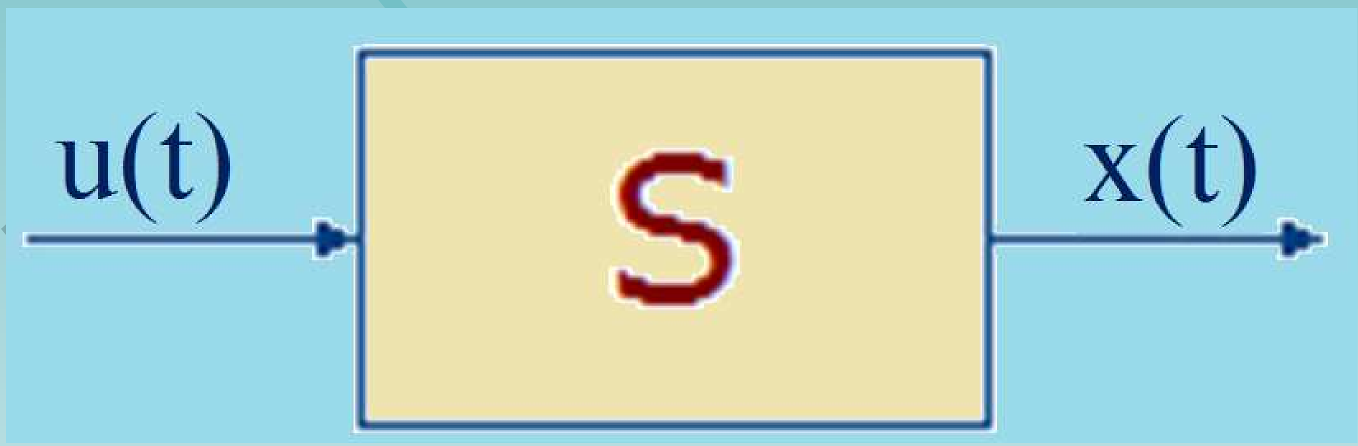
μ = 4 N·s/m

k = 3 N/m

spring / mass / damper

$m = 1$ kg $\qquad \mu = 4$ N·s/m $\qquad k = 3$ N/m

$$\begin{cases} m\dfrac{d^2x}{dt^2} + \mu\dfrac{dx}{dt} + k\,x = mx'' + \mu x' + kx = u, \\ x'(0) = a, \qquad x(0) = b \end{cases}$$

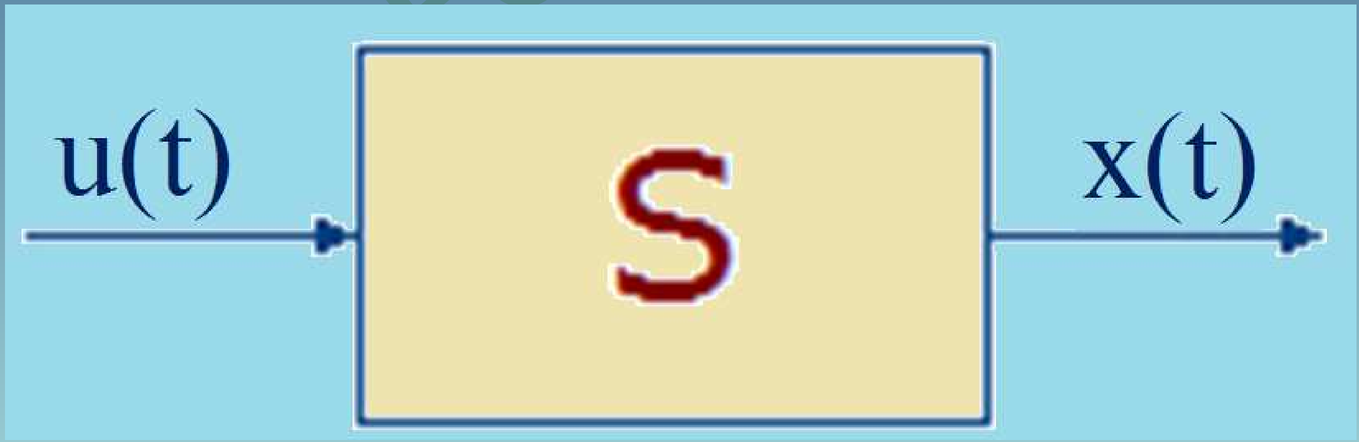


spring / mass / damper

m = 1 kg $\qquad \mu$ = 4 N·s/m $\qquad k$ = 3 N/m

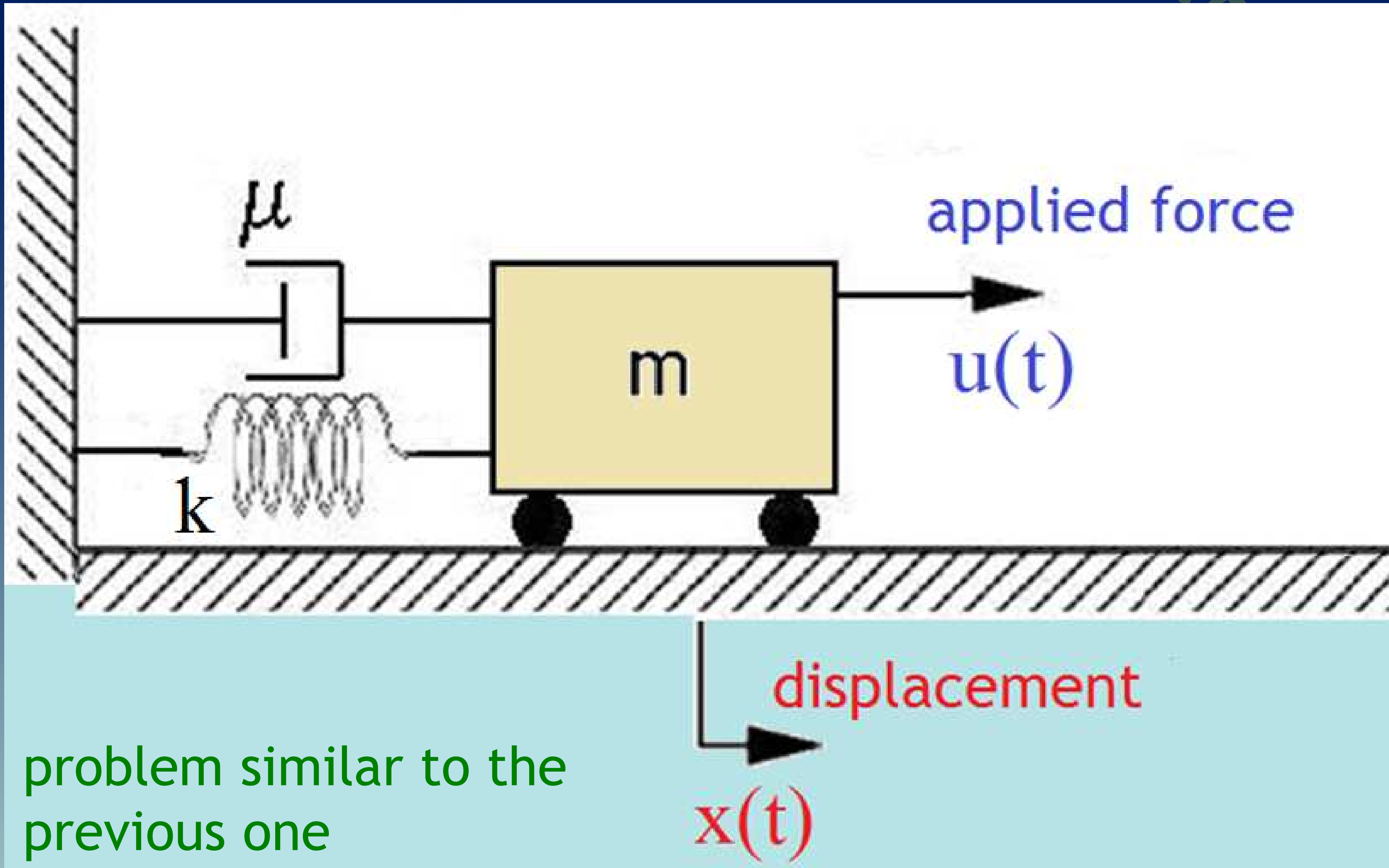
and the model becomes:

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = x'' + 4x' + 3x = u, \\ x'(0) = a, \qquad x(0) = b \end{cases}$$



translational mechanical motion

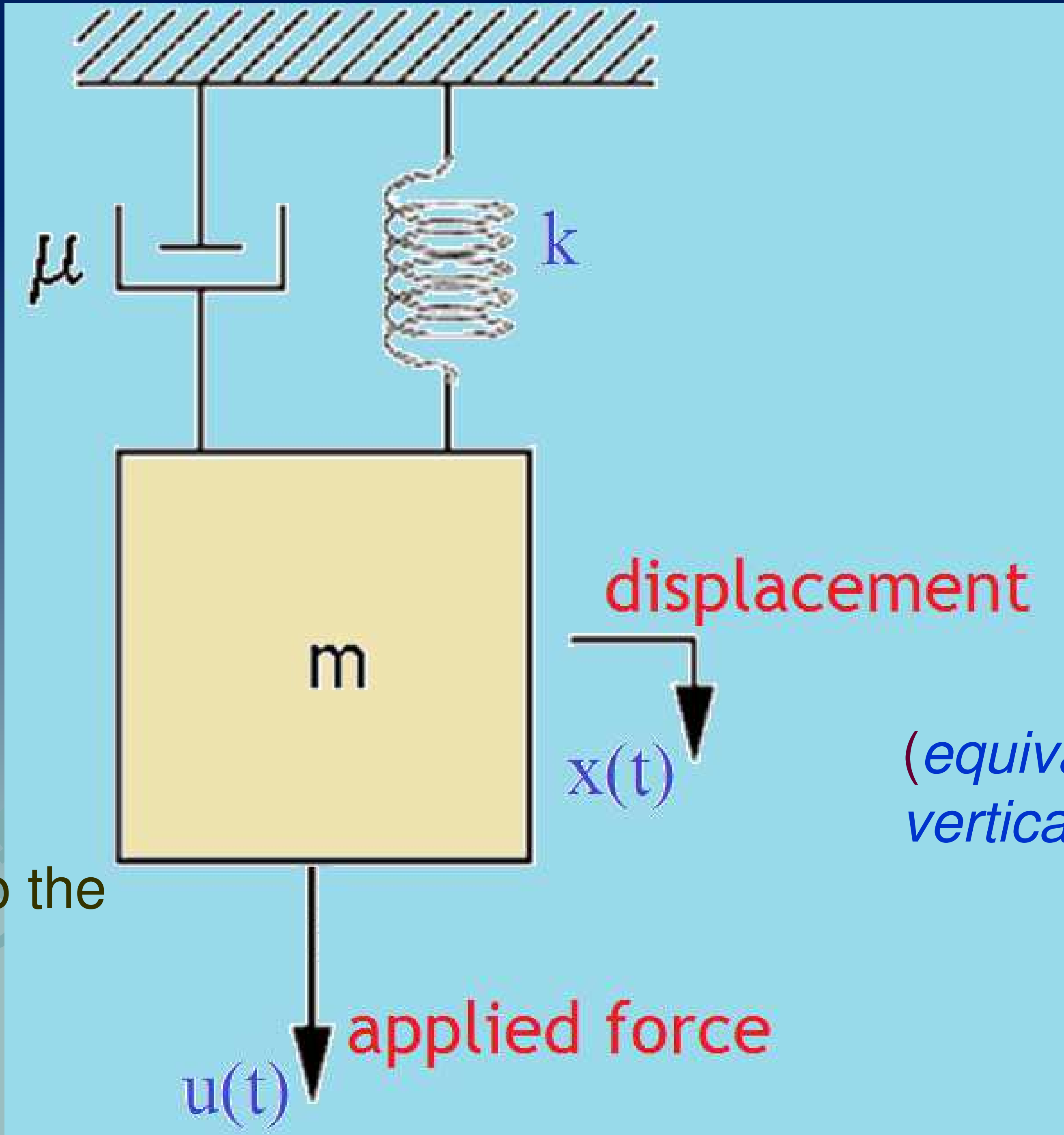
translational mechanical motion



problem similar to the previous one

spring / mass / damper

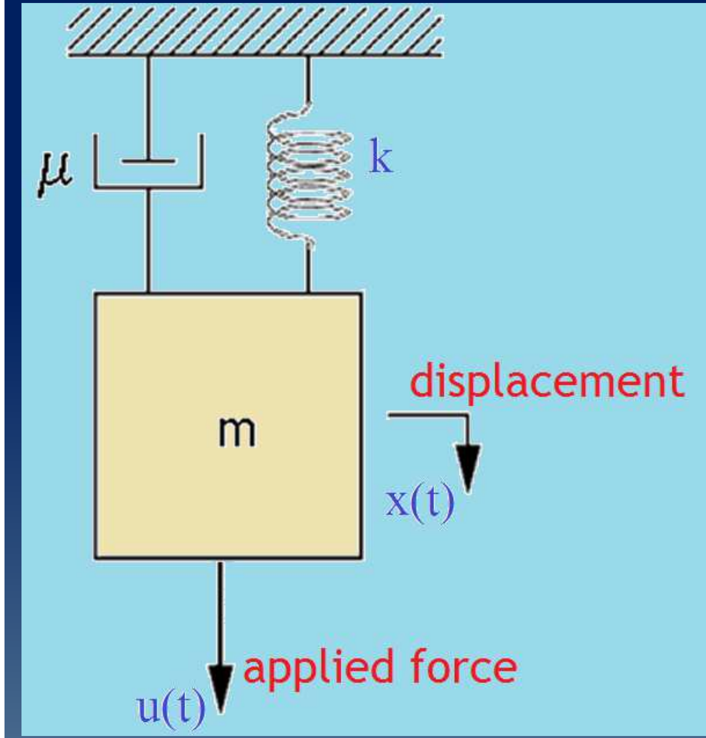
translational mechanical motion



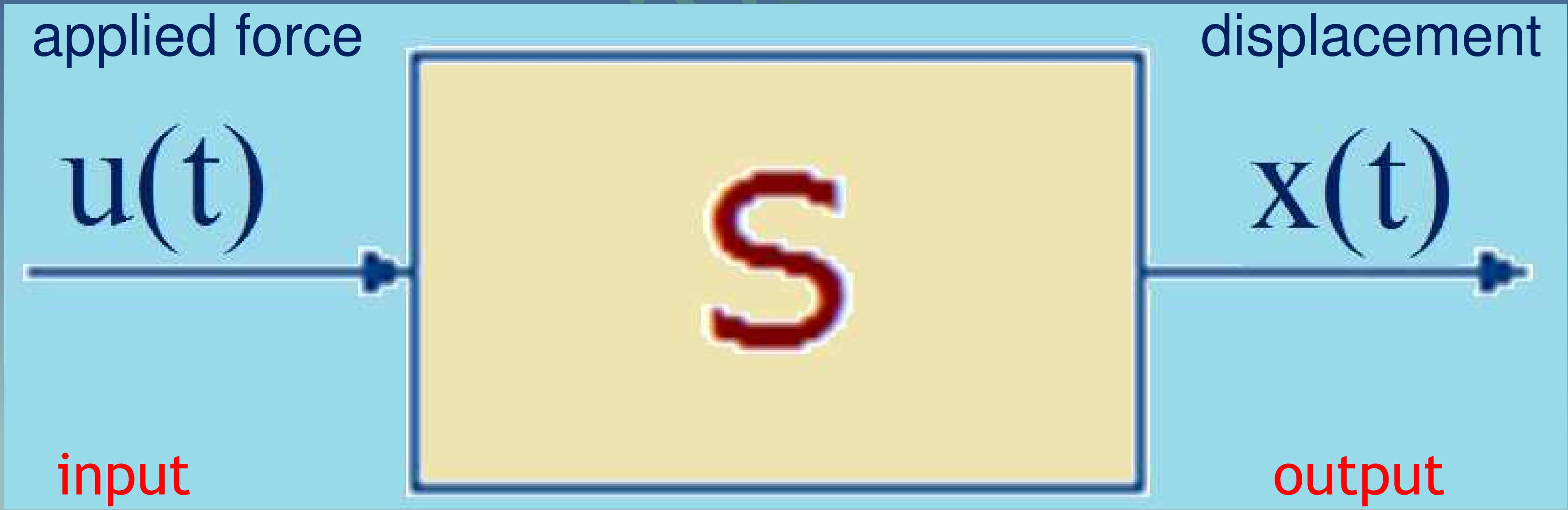
(*equivalent vertical*)

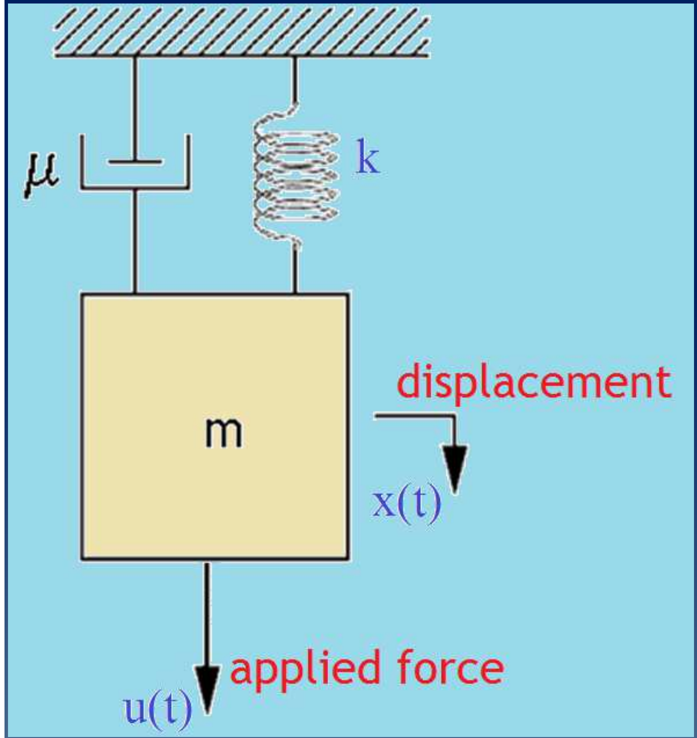
problem similar to the previous one
spring / mass / damper

Systems modelling



translational
mechanical
motion





translational
mechanical
motion

Again, using the Newton's 2nd Law we obtain:

$$m x'' + \mu x' + k x = u,$$

ou

$$m \frac{d^2 x}{dt^2} + \mu \frac{dx}{dt} + k\, x = u,$$

spring / mass / damper or translational mechanical motion

Thus, these two systems are described by the same *differential equation* (2nd order), that is, have the same model:

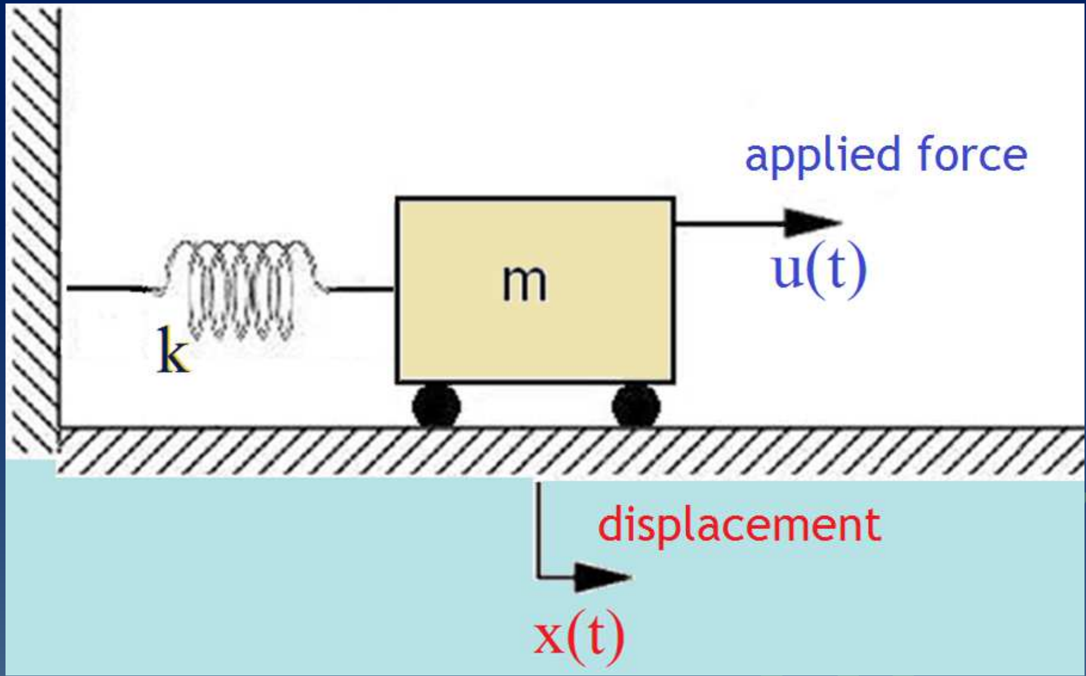
$$m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + k\,x =$$

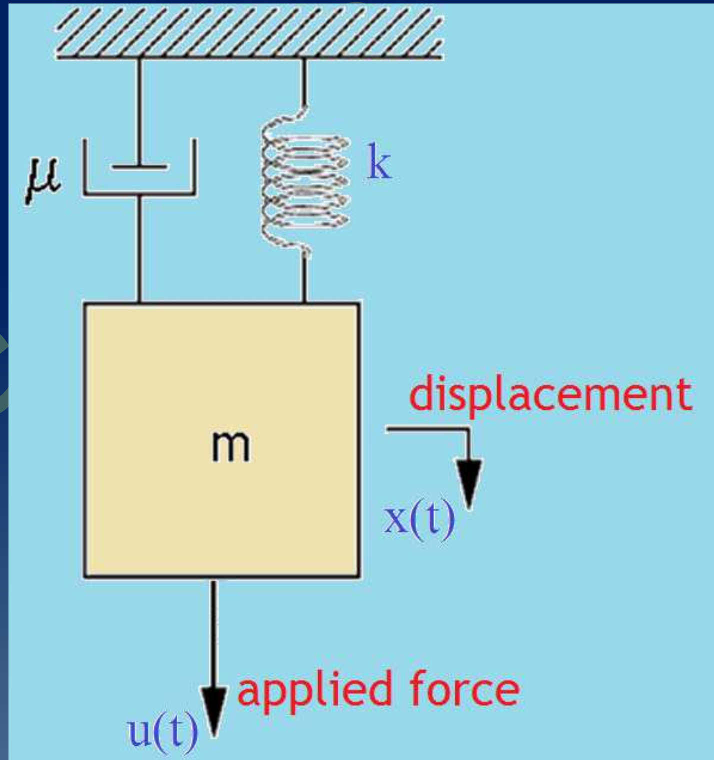
$$= mx'' + \mu x' + kx = u$$

initial conditions:

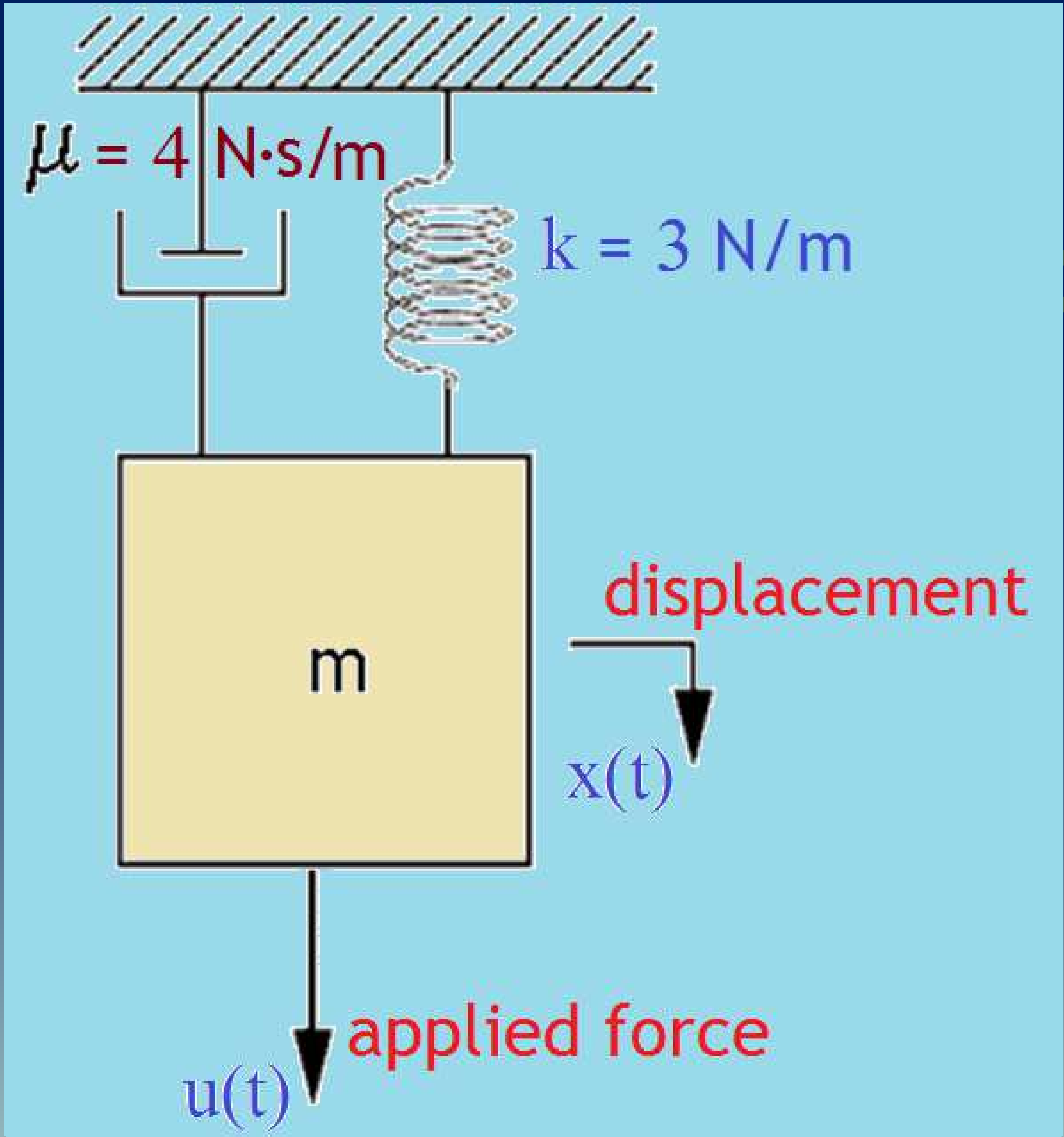
$$x'(0) = a, \qquad x(0) = b$$

spring / mass / damper or translational mechanical motion





$$\begin{cases} m\dfrac{d^2x}{dt^2} + \mu\dfrac{dx}{dt} + k\,x = mx'' + \mu x' + kx = u, \\ x'(0) = a, \qquad x(0) = b \end{cases}$$



Now, giving the same values to **m**, **μ** and **k** that has been given to the problem **spring / mass / damper**, we have:

$m = 1$ kg

$\mu = 4$ N·s/m

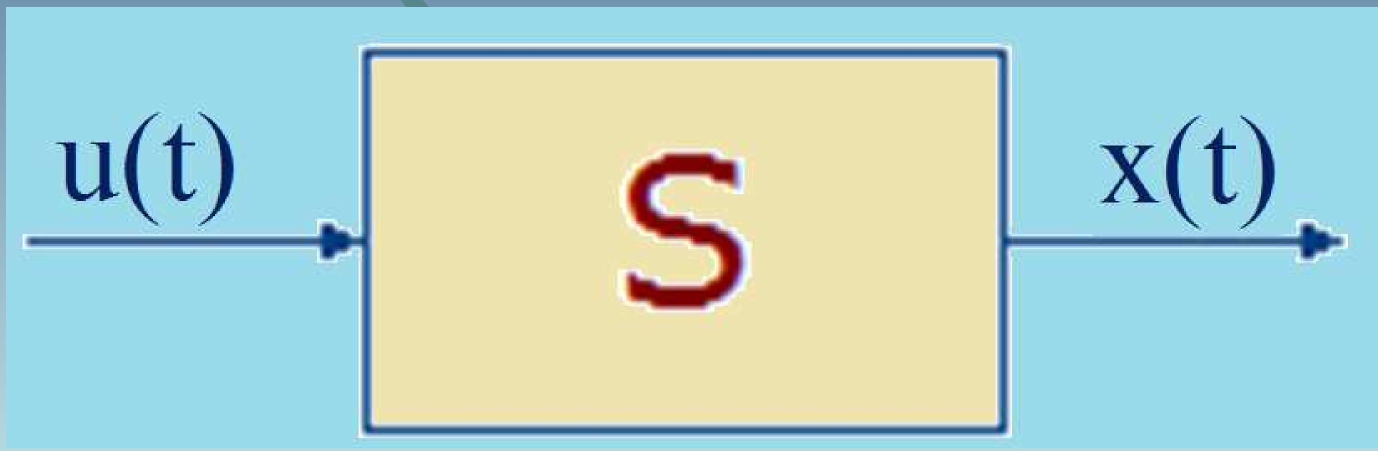
$k = 3$ N/m

Systems modelling

spring / mass / damper or translational mechanical motion

$m = 1$ kg $\qquad \mu = 4$ N·s/m $\qquad k = 3$ N/m

$$\begin{cases} m\dfrac{d^2x}{dt^2} + \mu\dfrac{dx}{dt} + k\,x = mx'' + \mu x' + kx = u, \\ x'(0) = a, \qquad x(0) = b \end{cases}$$

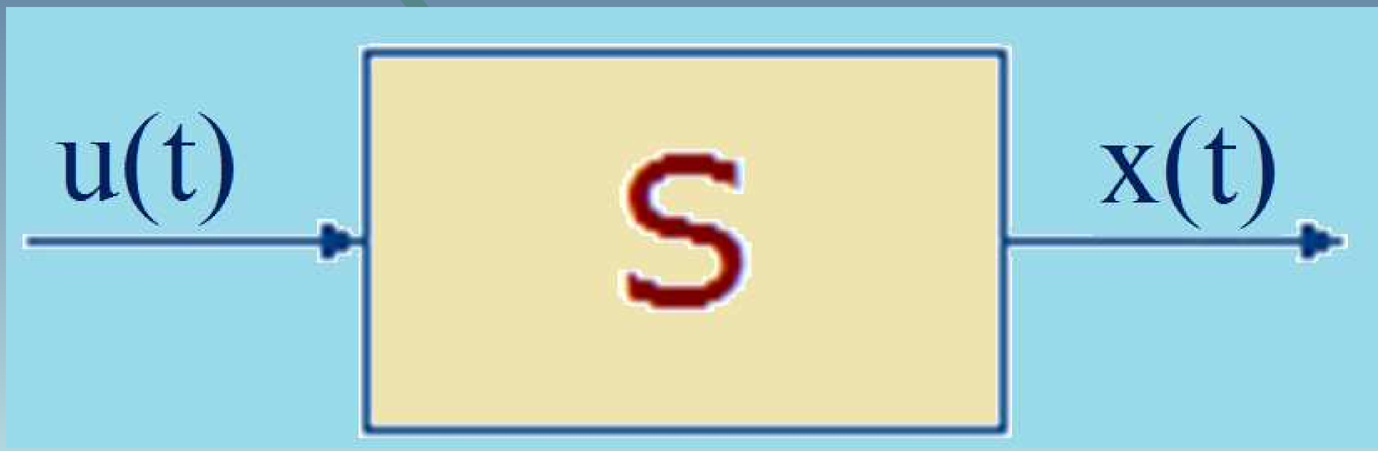


spring / mass / damper **or** translational mechanical motion

m = 1 kg μ = 4 N·s/m k = 3 N/m

(both have the same model)

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = x'' + 4x' + 3x = u, \\ \\ x'(0) = a, \qquad x(0) = b \end{cases}$$

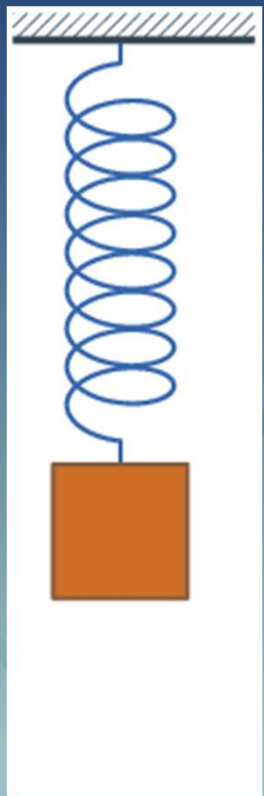


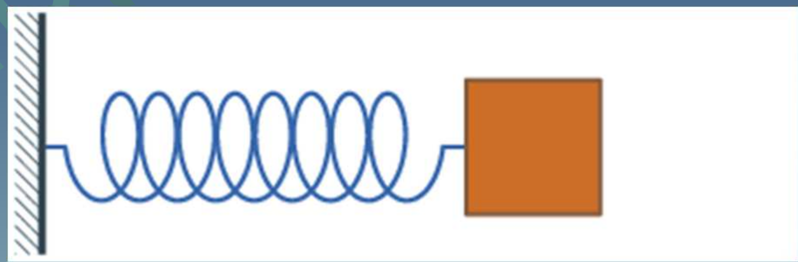
translational mechanical motion

Observation:

Note that if $\mu = 0$

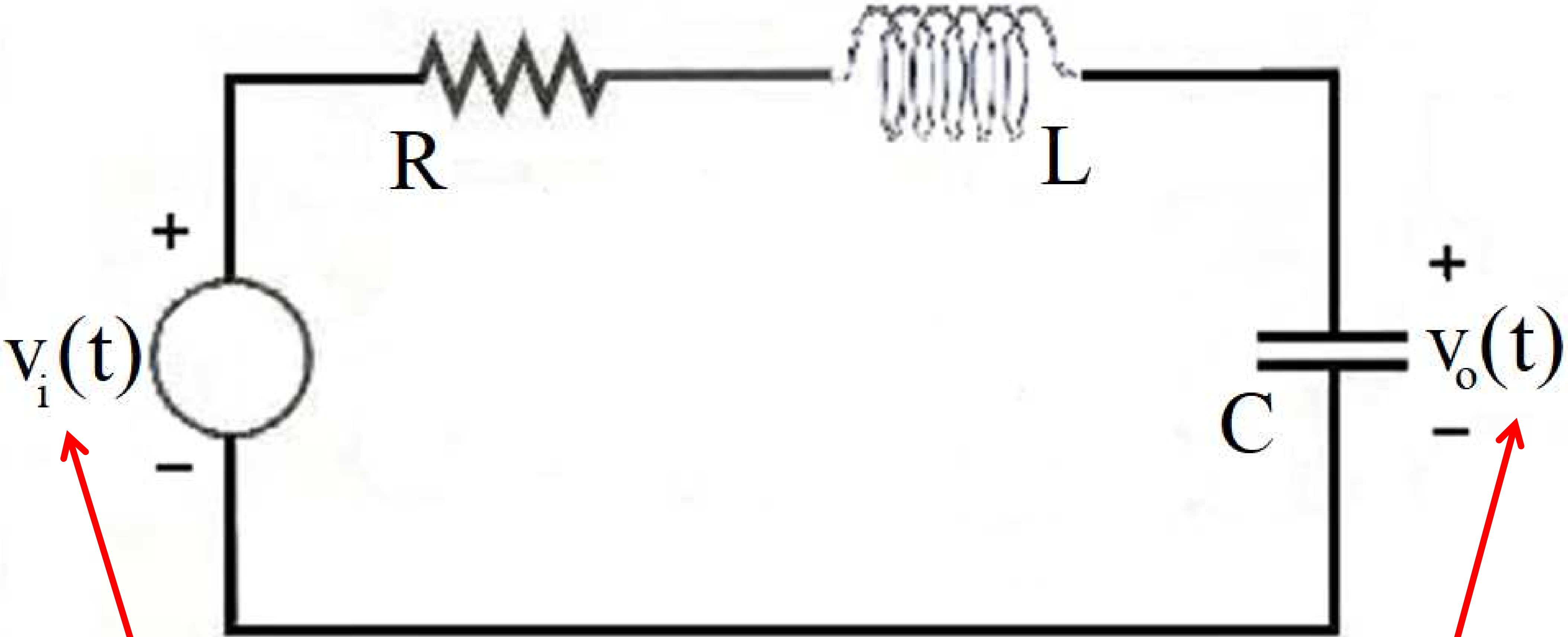
this system becomes the "*harmonic oscillator*".





circuito RLC série

RLC series circuit

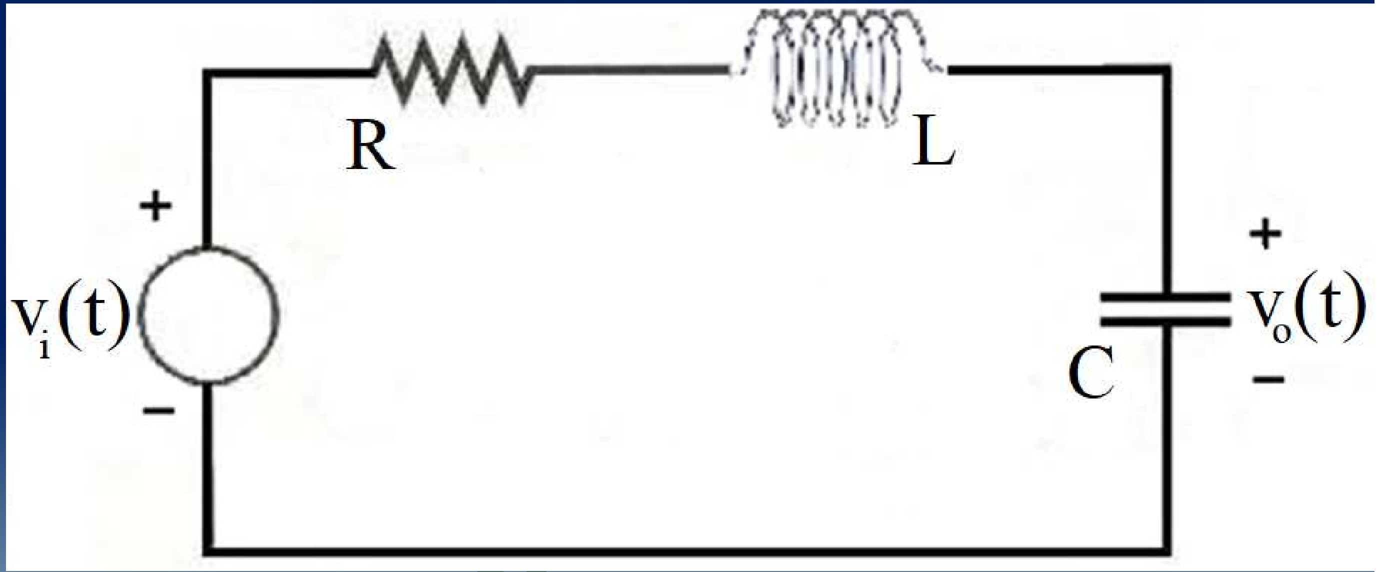


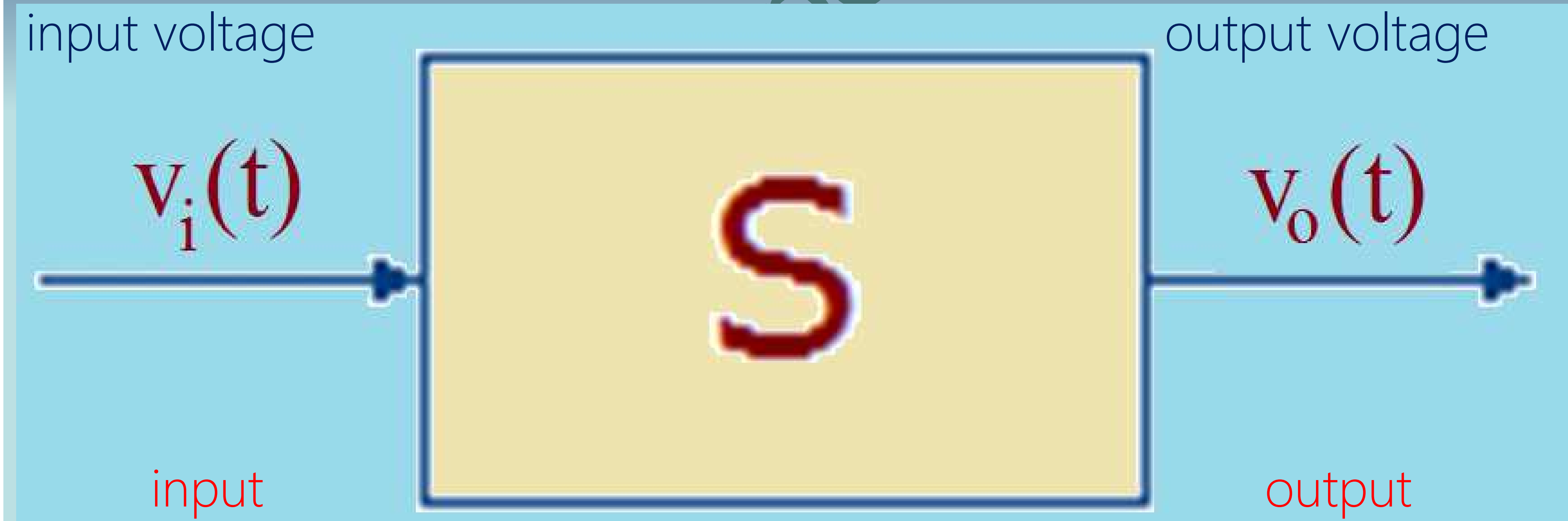
input
voltage

output
voltage

Systems modelling

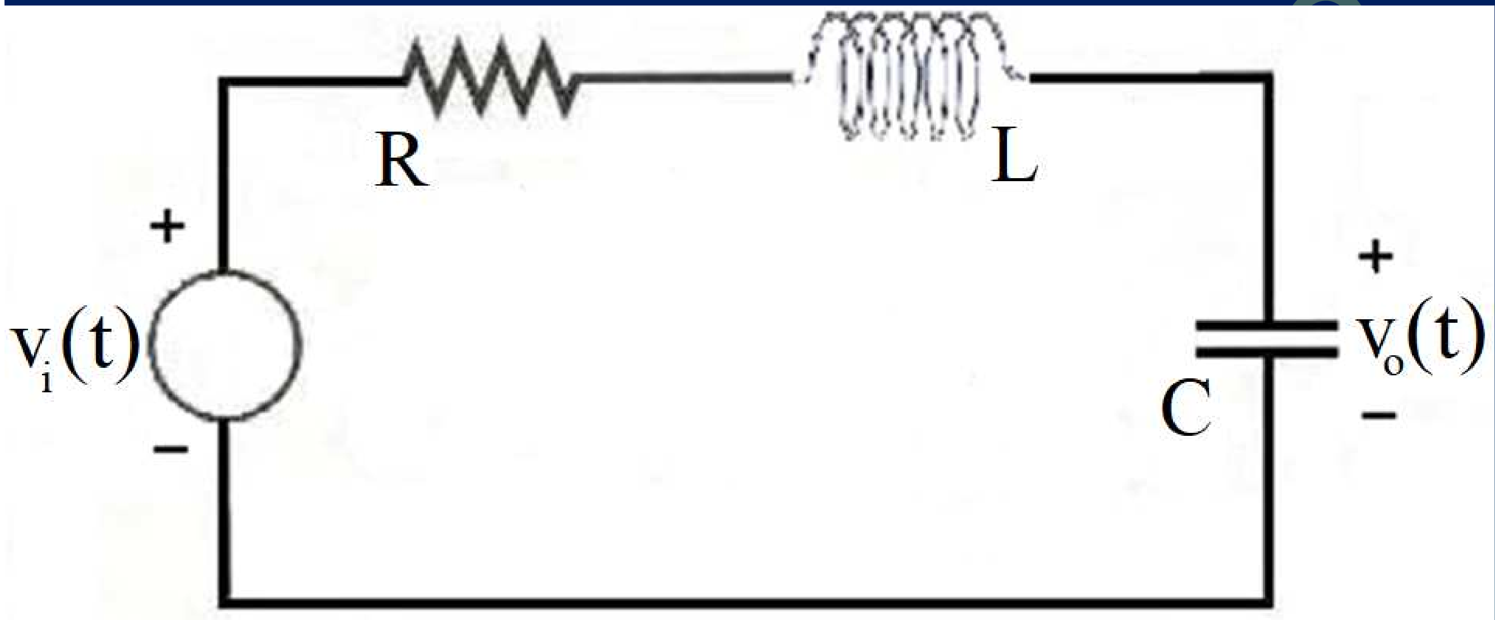
RLC series circuit

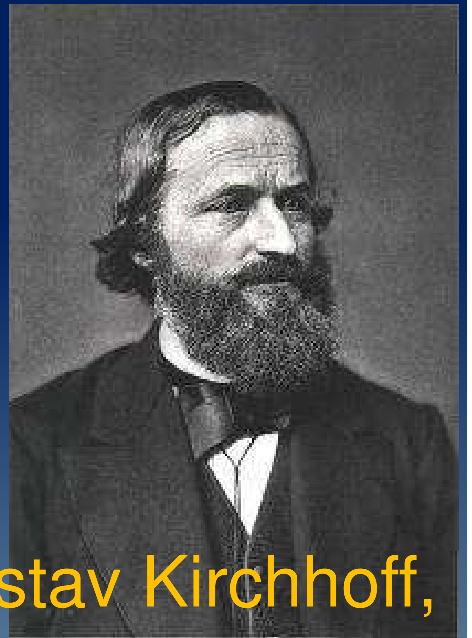




Kirchhoff Law (loop rule):

$$\sum v_k = 0$$





Gustav Kirchhoff, 1824-1887

thus

$$v_i - LC\,v_o'' - RC\,v_o' - v_o = 0,$$

RLC series circuit

and therefore,

$$LC\, v_o'' + RC\, v_o' + v_o = v_i \,,$$

or

$$LC \frac{d^2 v_o}{dt^2} + RC \frac{dv_o}{dt} + v_o = v_i \,,$$

Then, this *system* is also described by one *differential equation* of 2nd order, as the previous example.

RLC series circuit

That is, the model of this system is a *differential equation* of 2nd order:

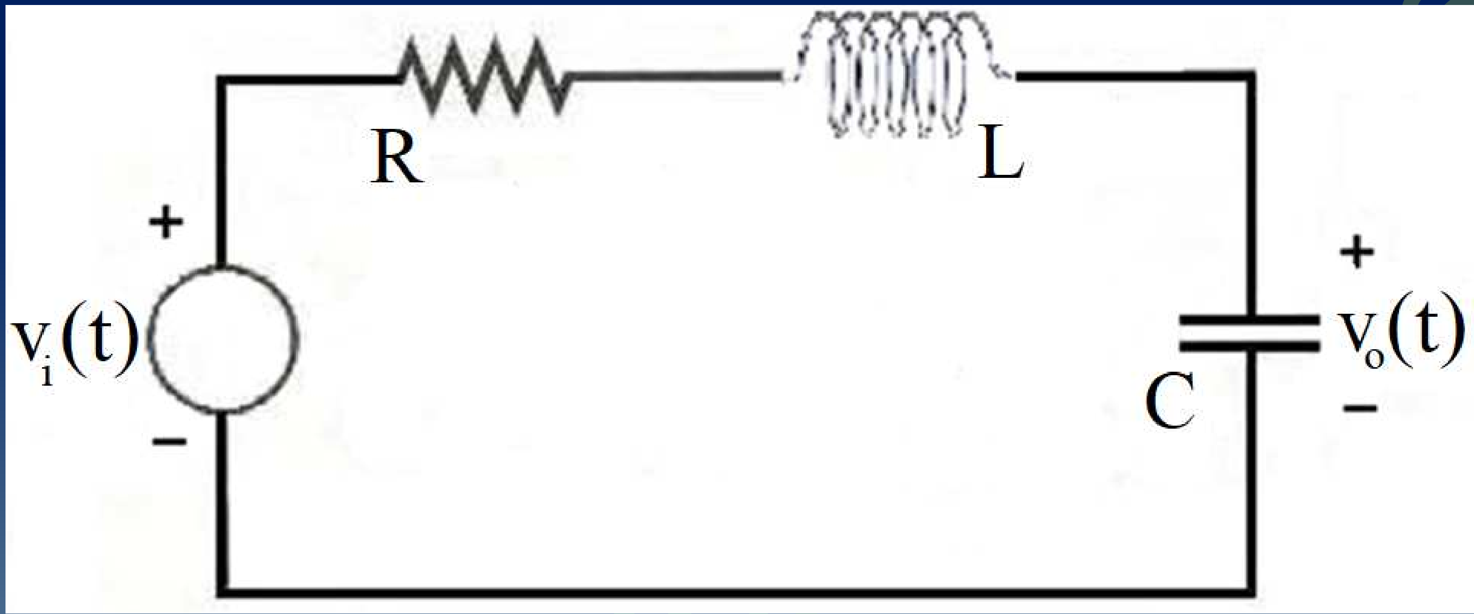
$$LC\frac{d^2v_o}{dt^2} + RC\frac{dv_o}{dt} + v_o =$$

$$= RC\,v_o'' + LC\,v_o' + v_o = v_i$$

initial conditions:

$$v_o'(0) = a, \qquad v_o(0) = b$$

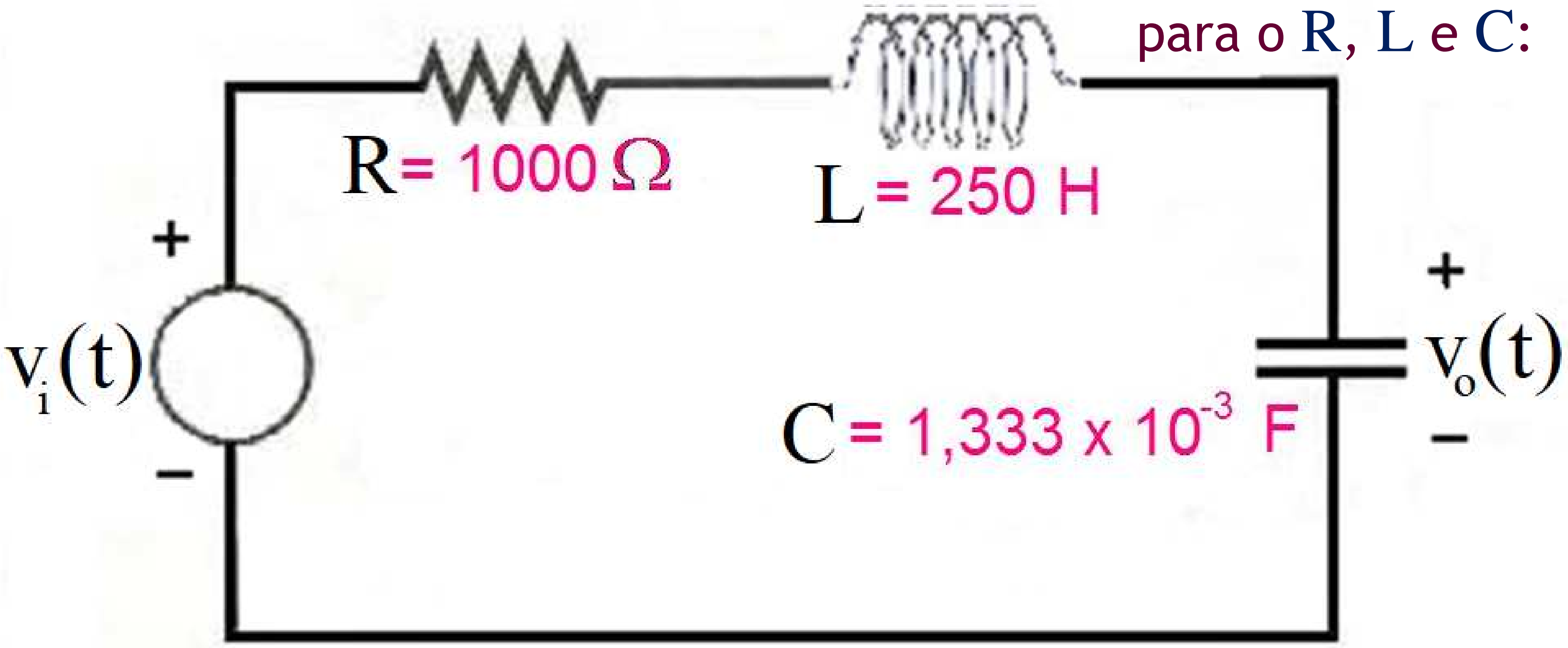
RLC series circuit



$$\begin{cases} LC\dfrac{d^2v_o}{dt^2} + RC\dfrac{dv_o}{dt} + v_o = LCv_o'' + RCv_o' + v_o = v_i, \\ v_o'(0) = a, \qquad v_o(0) = b \end{cases}$$

RLC series circuit

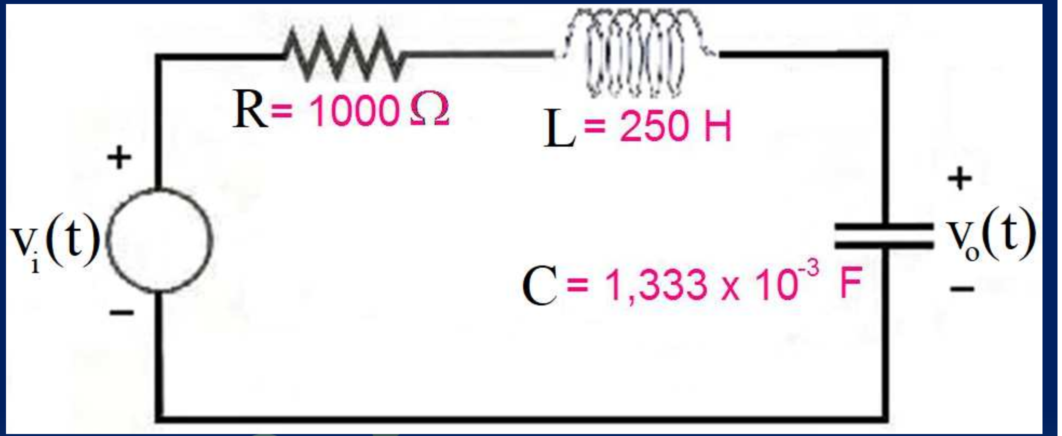
Dando valores para o R, L e C:



R = 1000 Ω

L = 250 H

C = $1{,}333 \times 10^{-3}$ F



RLC series circuit

$$\begin{cases} LC\dfrac{d^2v_o}{dt^2} + RC\dfrac{dv_o}{dt} + v_o = LCv_o'' + RCv_o' + v_o = v_i, \\ v_o'(0) = a, \qquad v_o(0) = b \end{cases}$$

R = 1000 Ω

L = 250 H

C = 1,333 x 10^{-3} F

RLC series circuit

$$\begin{cases} LC\dfrac{d^2v_o}{dt^2} + RC\dfrac{dv_o}{dt} + v_o = LCv_o'' + RCv_o' + v_o = v_i, \\ v_o'(0) = a, \qquad v_o(0) = b \end{cases}$$

R = 1000 Ω

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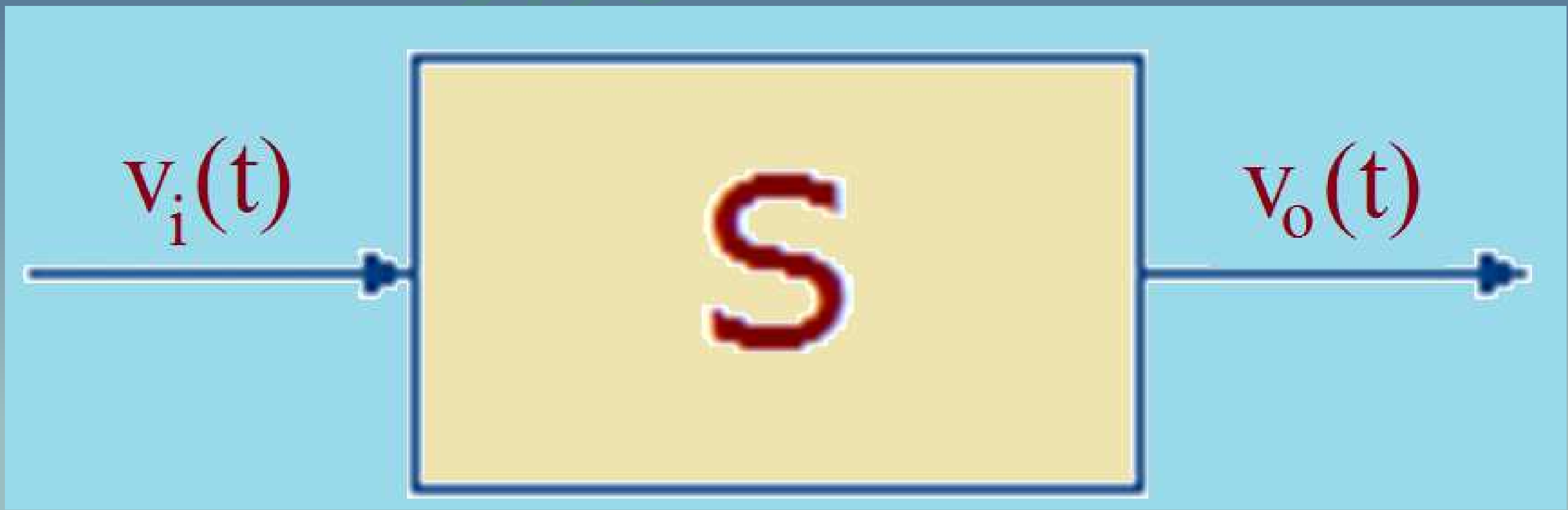


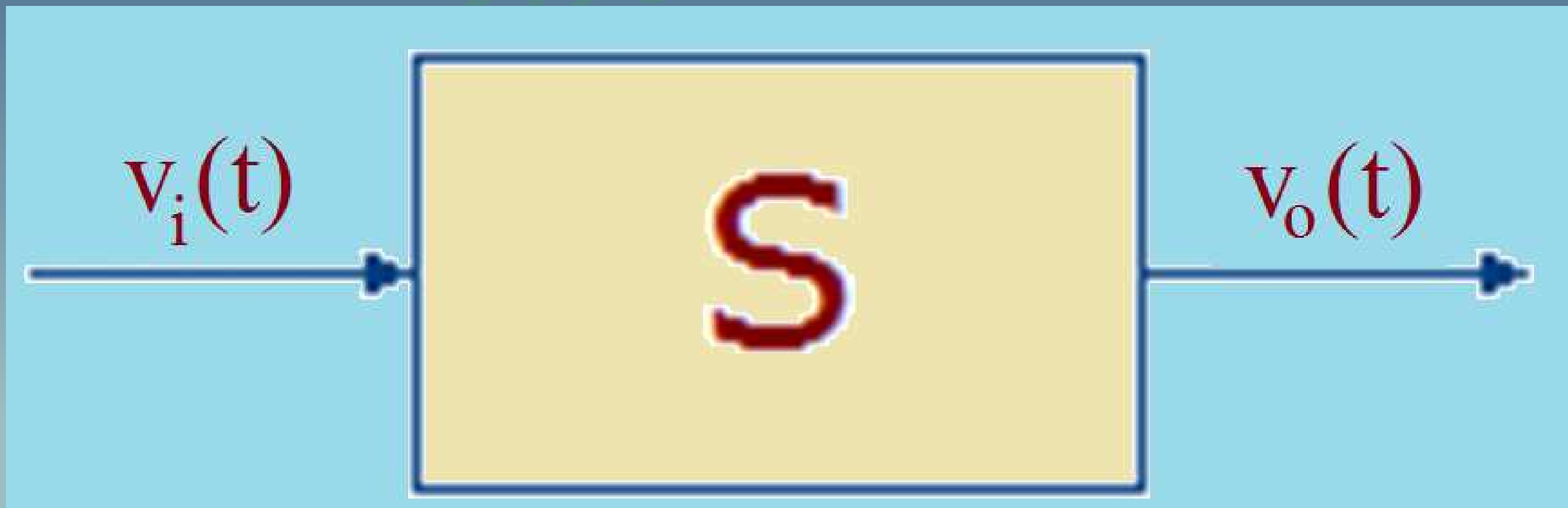
RLC series circuit

$$\begin{cases} \dfrac{d^2 v_o}{dt^2} + 4\dfrac{dv_o}{dt} + 3v_o = v_o'' + 4v_o' + 3v_o = 3v_i, \\ v_o'(0) = a, \qquad v_o(0) = b \end{cases}$$

R = 1000 Ω

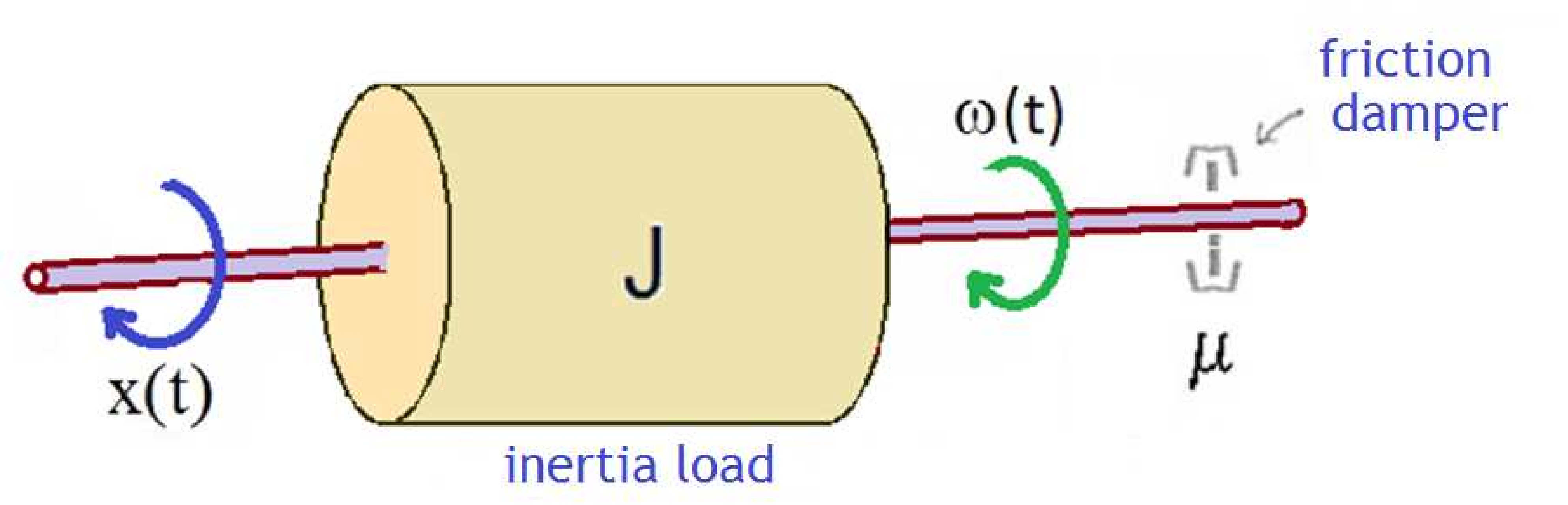
L = 250 H

C = 1,333 x 10^{-3} F



rotational mechanical motion

rotational mechanical motion



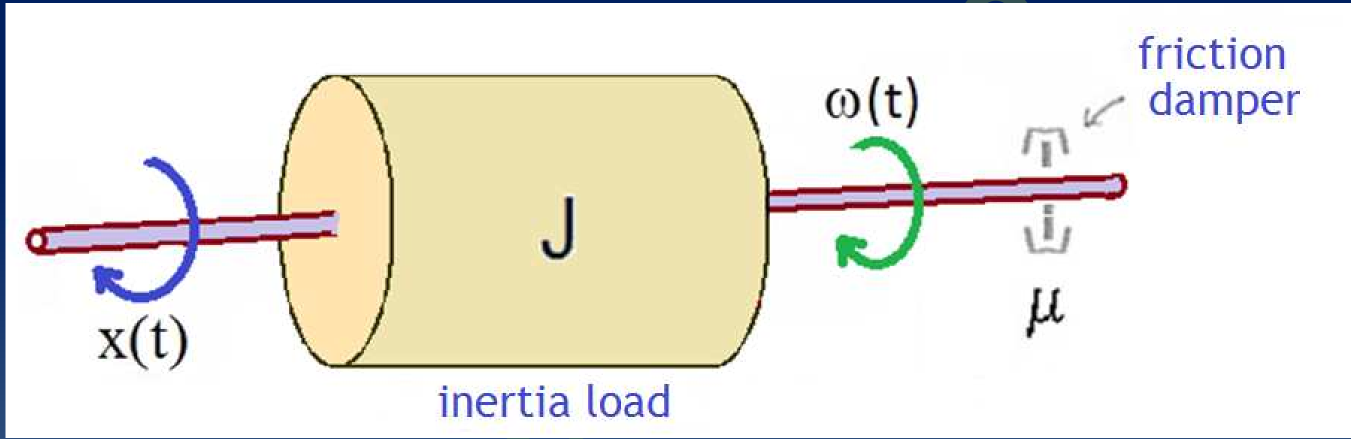
$x(t)$ = torque applied to the system input [N·m];

$\omega(t)$ = angular velocity *output* [rad/s];

J = moment of inertia [kg ·m^2];

μ = friction coefficient [N·m /rad/s]

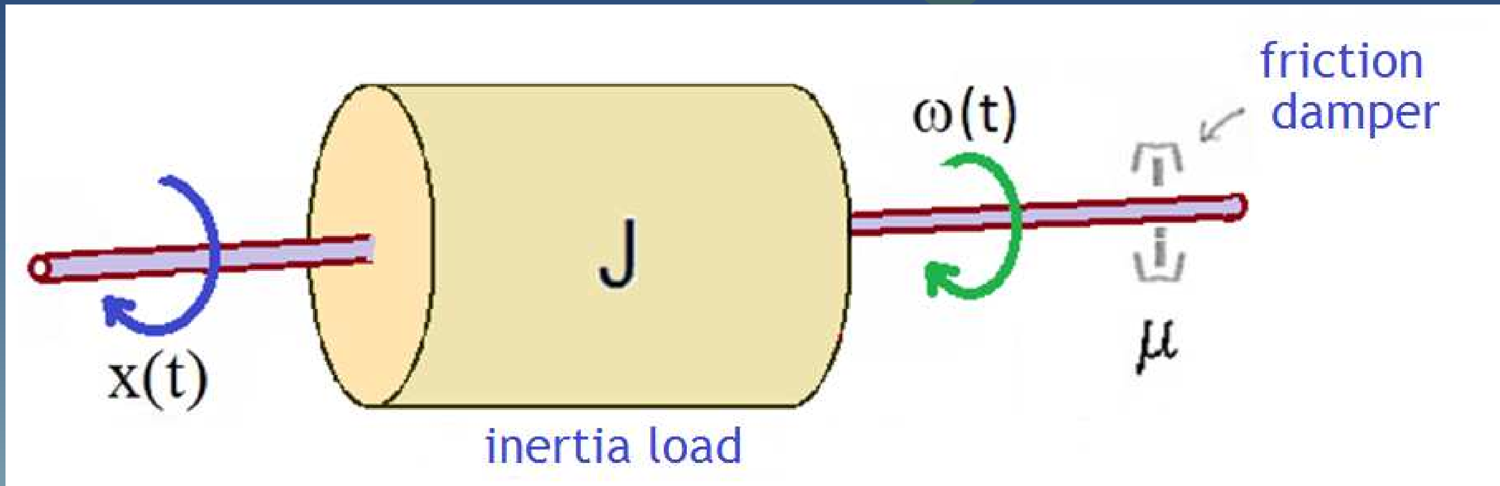
rotational mechanical motion



rotational mechanical motion

Using Newton's Law for rotational systems

$$\sum \text{momentos} = J\,\omega' ,$$



we obtain

$$J\,\omega' + \mu\,\omega = x ,$$

rotational mechanical motion

Thus, this system is described by a *differential equation* (of 1st order):

$$J\frac{d\omega}{dt} + \mu\,\omega =$$

$$= J\omega' + \mu\omega = x$$

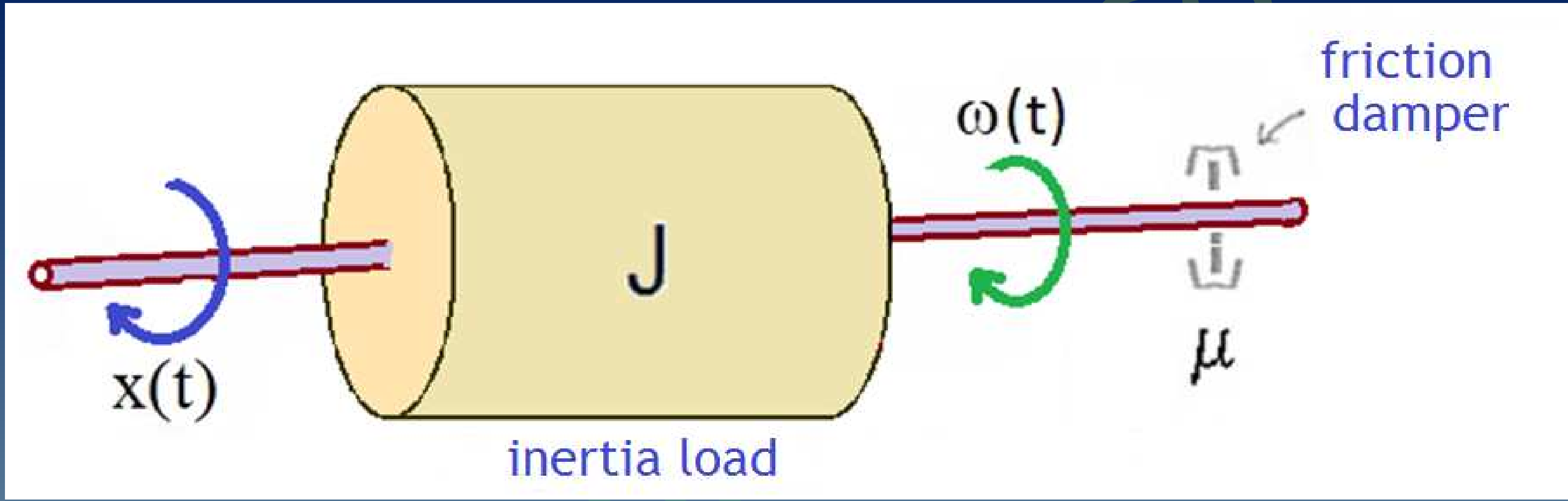
initial condition:

$$\omega(0) = a$$

rotational mechanical motion

that is,

the model of this system is a *differential equation* of 1st order:



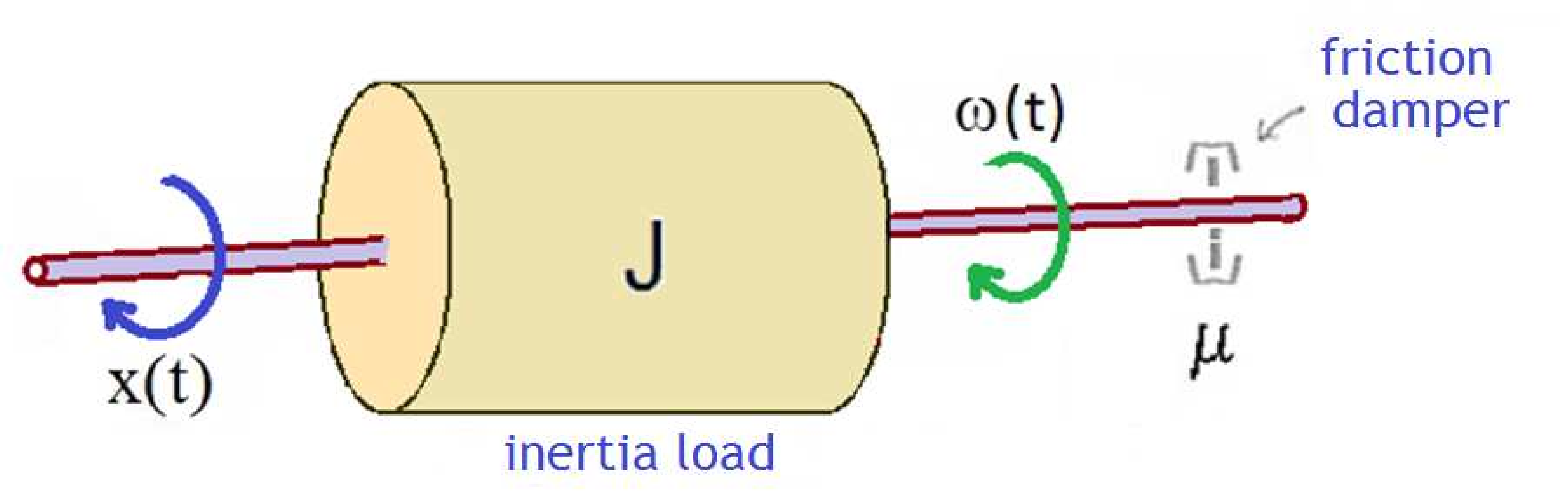
$$\begin{cases} J\dfrac{d\omega}{dt} + \mu\omega = J\omega' + \mu\omega = x\ , \\ \omega(0) = a \end{cases}$$

rotational mechanical motion

$J = 0{,}5$ kg/m^2

Now, giving values to J and μ:

$\mu = 2$ N·m /rad/s



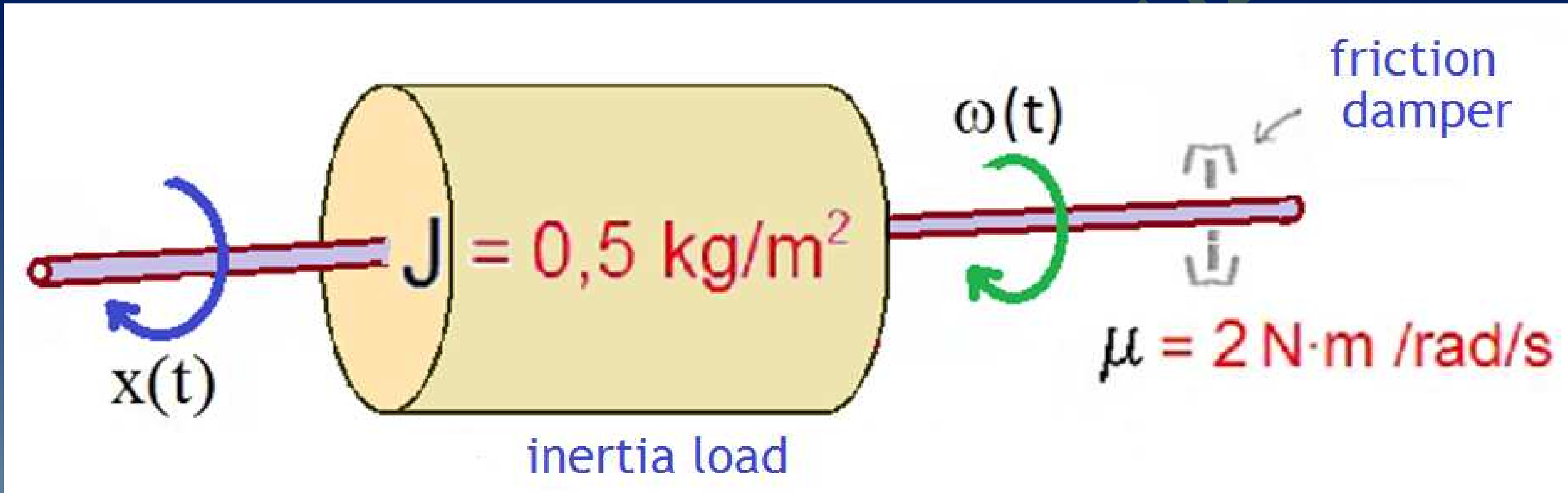
$$\begin{cases} J\dfrac{d\omega}{dt} + \mu\omega = J\omega' + \mu\omega = x, \\ \omega(0) = a \end{cases}$$

rotational mechanical motion

$J = 0{,}5\ kg/m^2$

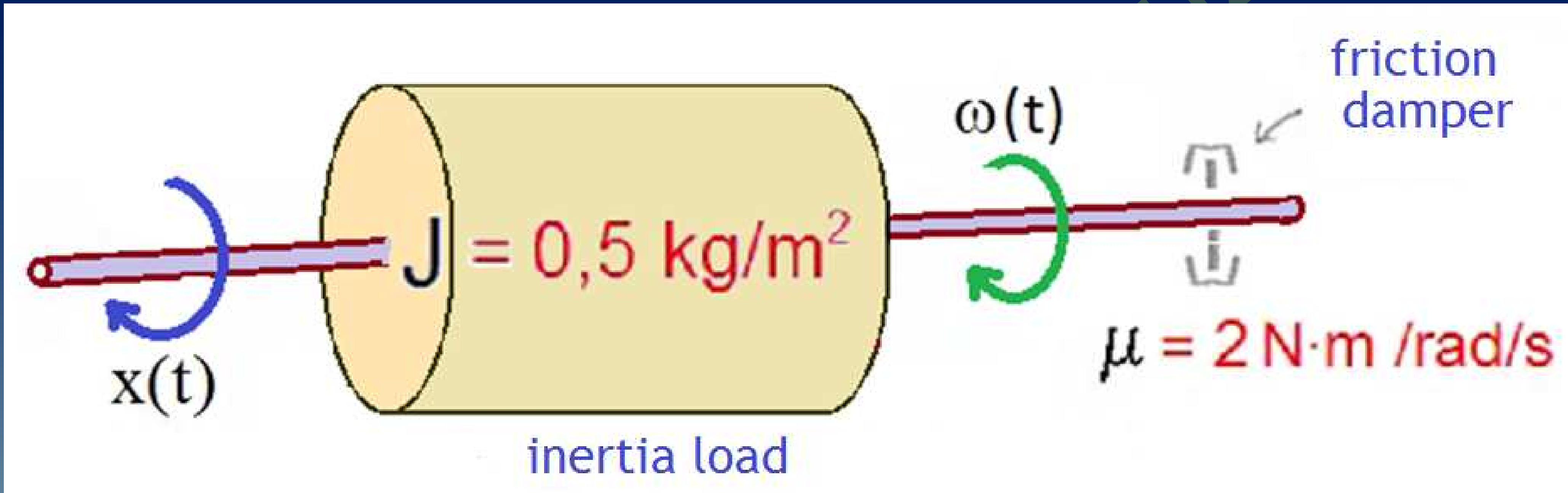
Now, giving values to J and μ:

$\mu = 2\ N \cdot m\ /rad/s$



$$\begin{cases} J\dfrac{d\omega}{dt} + \mu\omega = J\omega' + \mu\omega = x \\ \omega(0) = a \end{cases}$$

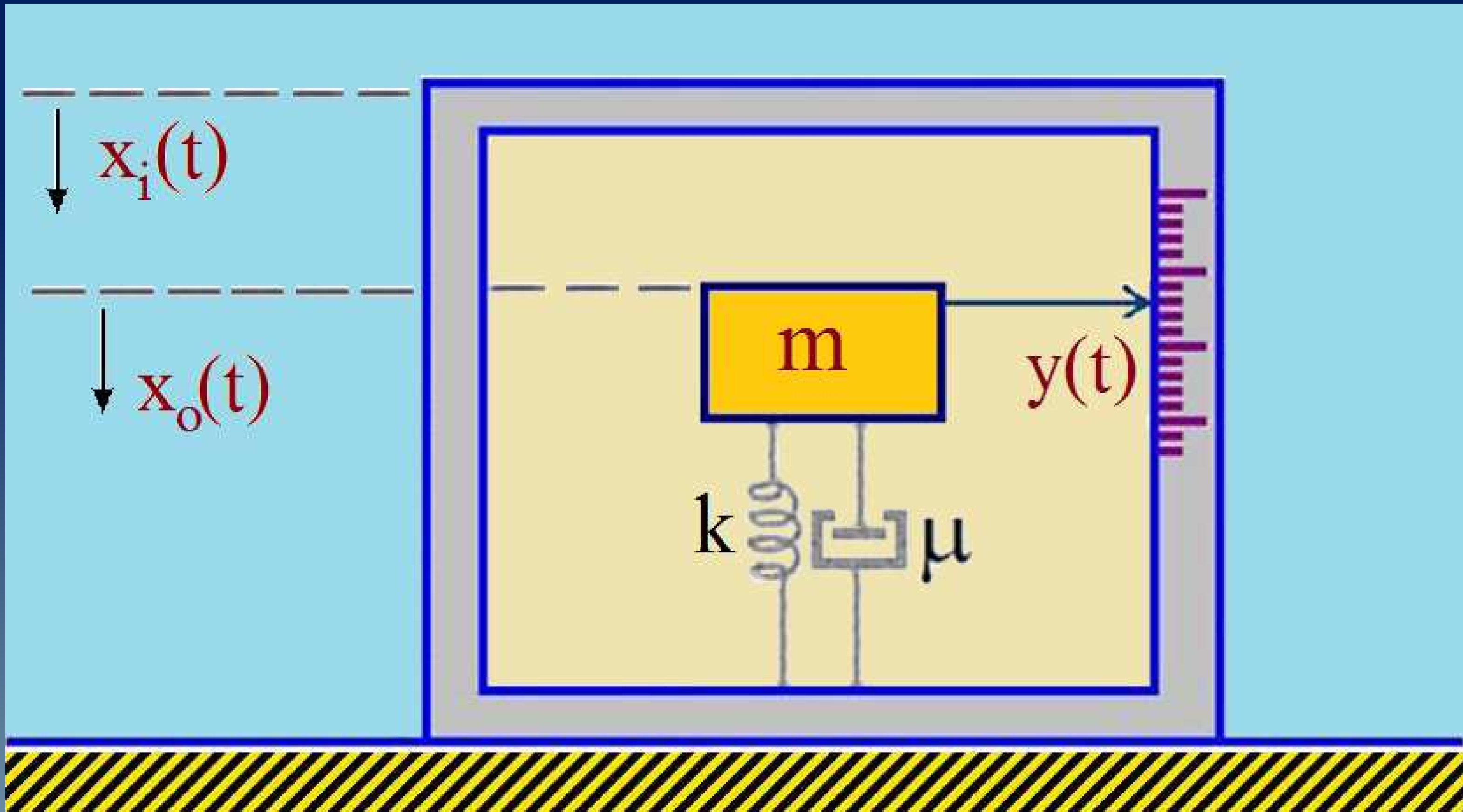
rotational mechanical motion



$$\begin{cases} \dfrac{d\omega}{dt} + 4\,\omega = \omega' + 4\,\omega = 2\,x \\ \omega(0) = a \end{cases}$$

a seismograph

seismograph



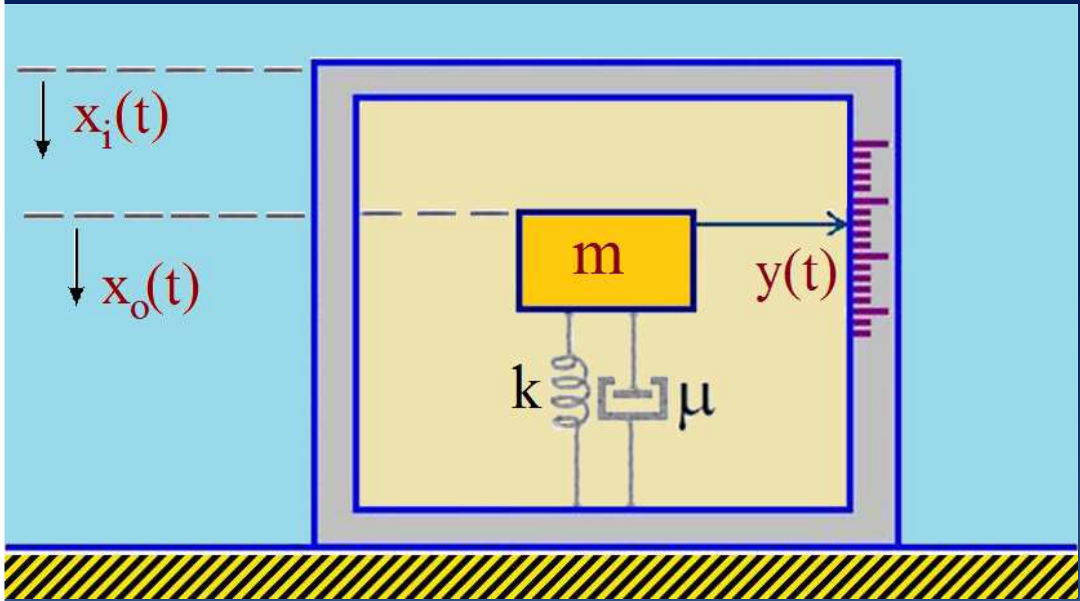
$x_i(t)$ = box *displacement* with respect to *inertial space;*

$x_o(t)$ = mass *displacement* with respect to *inertial space;*

$y(t)$ = mass *displacement* with respect to the *box.*

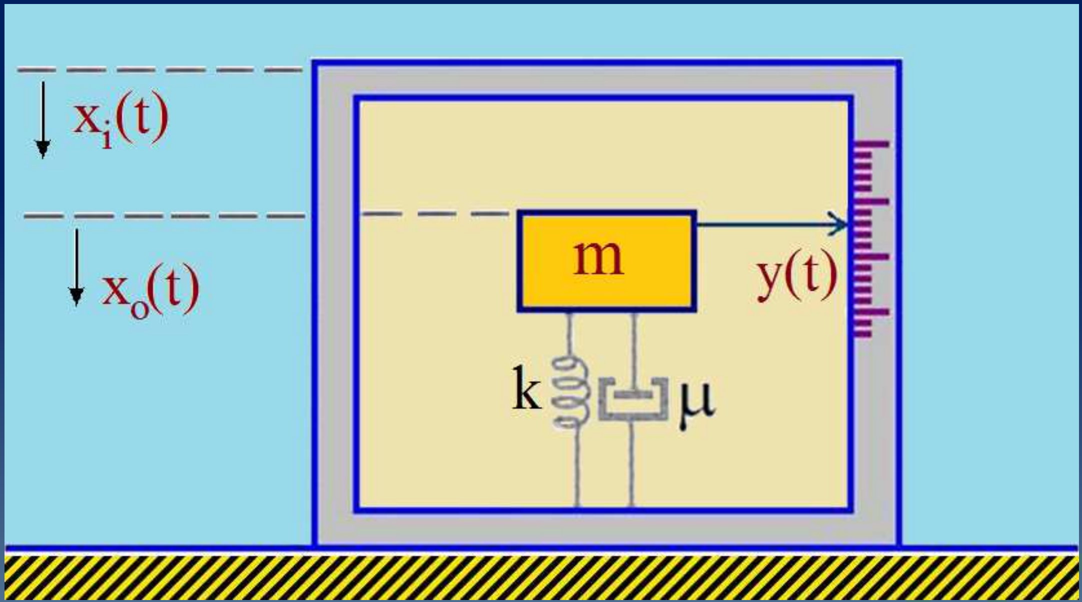
$$y(t) = [x_o(t) - x_i(t)]$$

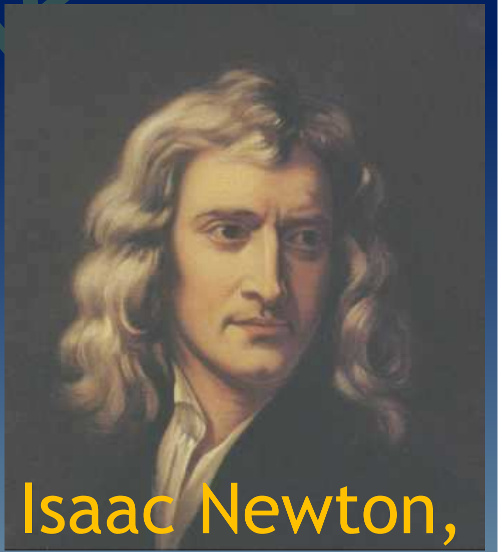
Systems modelling



seismograph

Again, by Newton's 2nd Law





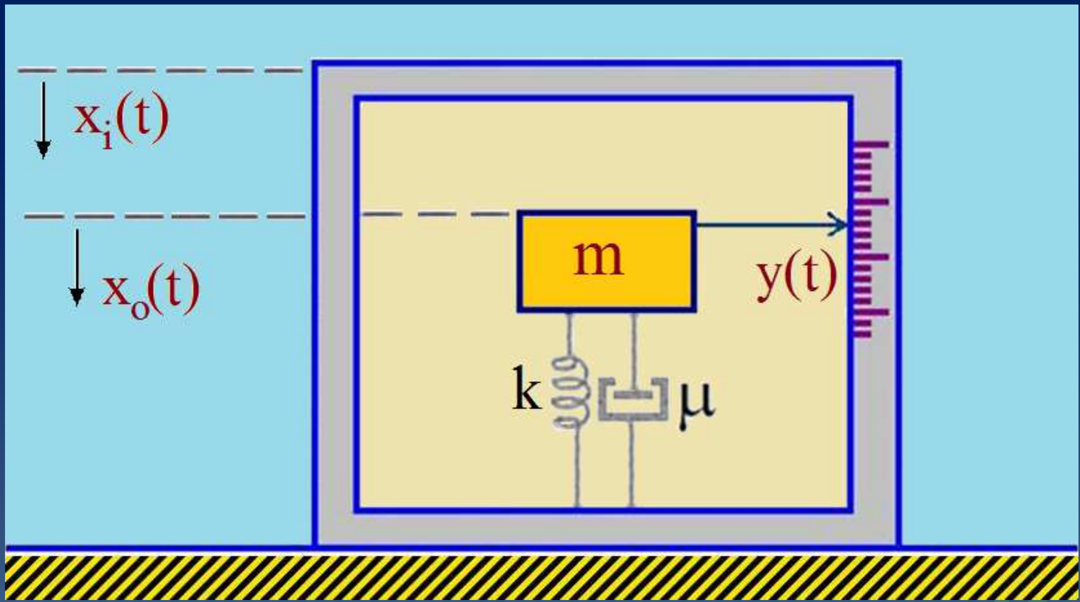
Sir Isaac Newton, 1643-1727

$$m x_o'' = -\mu\,(x_o' - x_i') - k(x_o - x_i),$$

and therefore,

$$m\underbrace{(x_o'' - x_i'')}_{y''(t)} + \mu\,\underbrace{(x_o' - x_i')}_{y'(t)} + k\underbrace{(x_o - x_i)}_{y(t)} = -m\,x_i'',$$

Systems modellling



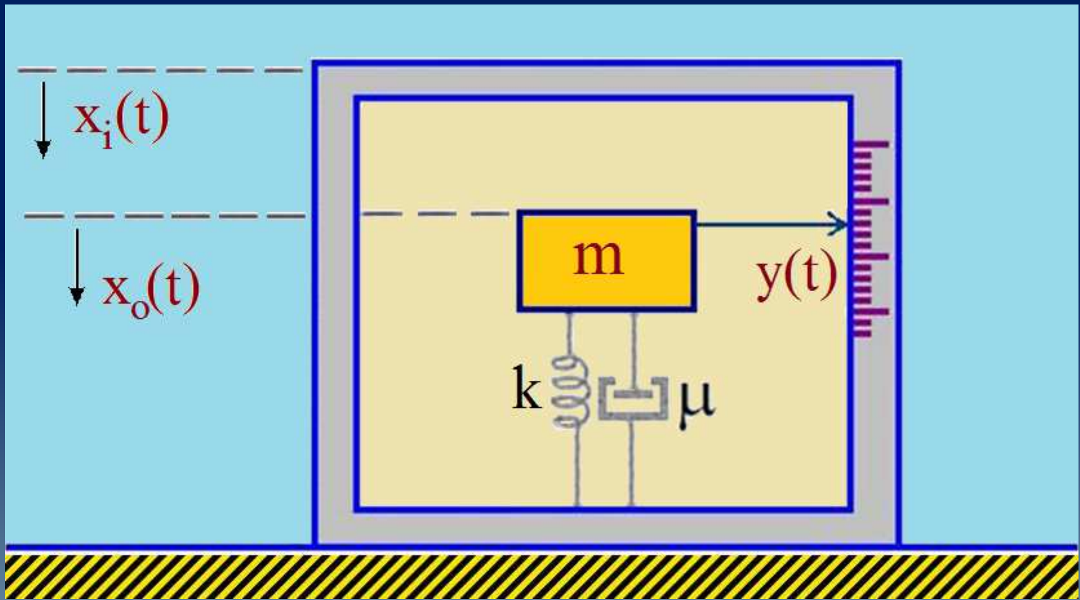
seismograph

thus,

$$m\,y'' + \mu\,y' + k\,y = -m\,x_i'',$$

or,

$$m\frac{d^2y}{dt^2} + \mu\frac{dy}{dt} + k\,y = -m\frac{d^2x_i}{dt^2},$$



seismograph

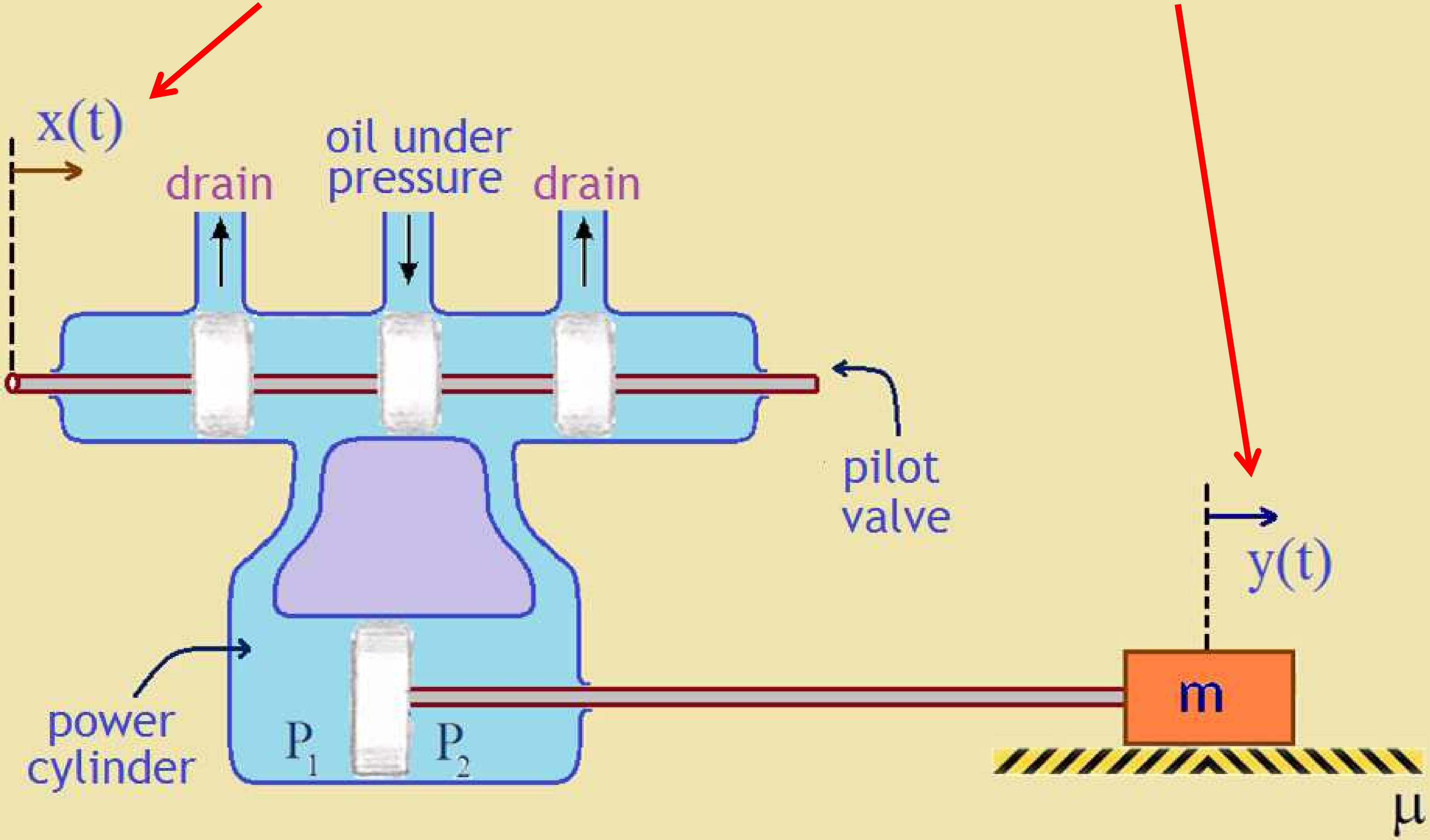
$$\begin{cases} m\dfrac{d^2y}{dt^2} + \mu\dfrac{dy}{dt} + ky = \\ \qquad = m\,y'' + \mu\,y' + k\,y = -m\,x_i'' , \\ y(0) = a\,, \quad y'(0) = b \end{cases}$$

a hydraulic servo-motor

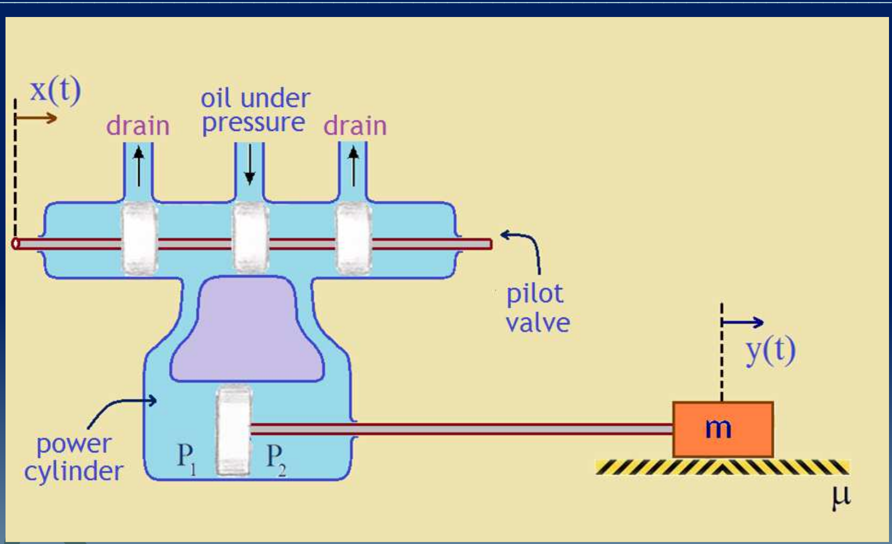
hydraulic servo-motor

input of the system

output of the system



hydraulic servo-motor



we obtain,

$$m\,y''(t) + \left(\mu + \frac{A^2\rho}{K_2}\right) y'(t) = \frac{AK_1}{K_2}\,x(t)\,,$$

or,

$$m\frac{d^2y}{dt^2} + \left(\mu + \frac{A^2\rho}{K_2}\right)\frac{dy}{dt} = \frac{AK_1}{K_2}\,x(t)\,,$$

hydraulic servo-motor

$$m\,y''(t) + \left(\mu + \frac{A^2\rho}{K_2}\right) y'(t) = \frac{AK_1}{K_2}\,x(t)\,,$$

A = piston area [m²];

ρ = oil density [kg/m³];

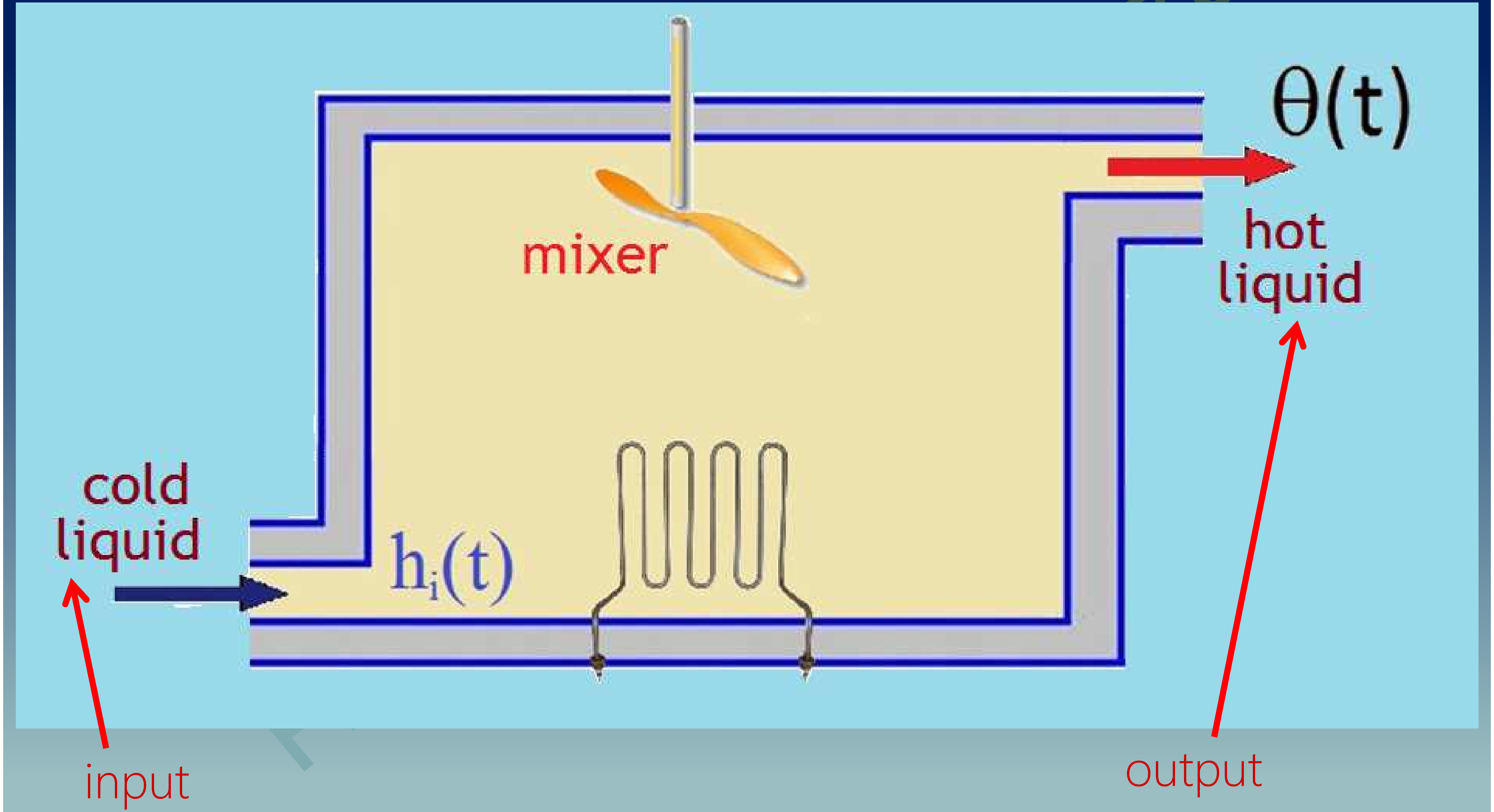
Q = flow rate of oil that goes to the power cylinder (*mass flow rate*) [kg/s];

$\Delta P = (P_1 - P_2)$ = pressure difference in the power cylinder (*pressure drop*) [N/m²].

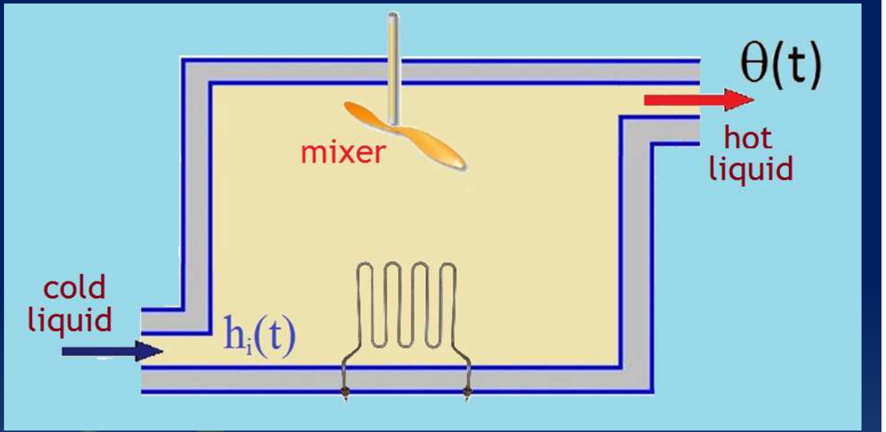
$$Q = K_1 \cdot x - K_2 \cdot \Delta P$$

a thermal system

thermal system



thermal system



we obtain,

$$RC\frac{d\theta}{dt} + \theta(t) = R\,h_i(t)\,,$$

where,

$h_i(t)$ = heat input rate [cal/s];

$\theta(t)$ = heat output temperature [ºC];

R = thermal resistance (*gain of the system*) [ºC·s/cal];

C = thermal capacitance (*heat capacity*) [cal/ºC];

T = **RC** = time constant of the system [s].

These examples above are simple, but serve to illustrate that many physical systems can be modeled as ordinary differential equations.

Therefore, many of the differential equations we have already seen in examples in the previous chapter may be the modeling of an original physical system.

other examples

A system described by partial differential equations:

$$\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) =$$

$$= k\,(u_{xx} + u_{yy} + u_{zz})$$

this system describes the *space wave propagation*

system linear
continuous,
time invariant,
with memory
and causal

Systems modelling

or, a system described by
difference equations:

$$y[n] = 2\left(x[n]\right)^2 - 4\,n\,x[n]$$

discrete system,
nonlinear,
time variant,
without memory
and causal.

More *complex systems* are represented not only by one, but by several equations.

$$\begin{cases} \dot{x}_1 = \dfrac{s\theta}{\theta + x_4} + \lambda(x_1, x_2, x_3)\, x_1 - x_1\left[\mu_1 + k_1(m_1)\, x_4\right] \\ \dot{x}_2 = \omega\, k_1(m_1)\cdot x_4 x_1 - x_2\left[\mu_2 + k_{20}\right] \\ \dot{x}_3 = (1-\omega)\, k_1(m_1)\, x_4\, x_1 + k_{20}\, x_2 - \mu_3\, x_3 \\ \dot{x}_4 = N(t)\,\mu_3\, x_3 - x_4\left[k_1(m_1)\cdot x_1 + \mu_v\right] \end{cases}$$

This system describes the dynamic of the evolution of *AIDS*

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Thank you!

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