NONLINEAR CONTROL BASED ON DIFFERENTIAL GEOMETRY

Gildeberto de Souza Cardoso†, Leizer Schnitman†, J.A.M. Felippe de Souza††, Oswaldo Ludwig Jr†
†Federal University of Bahia, Rua Aristides Novis, nº 2, Federação, CEP: 40.210-630, Salvador-BA, Brazil.
††Beira Interior University, Covilhã, Portugal.
E-mails: gildeberto@yahoo.com.br; leizer@ufba.br; fellipe@ubi.pt; oludwig@ig.com.br

ABSTRACT
This paper presents a study on nonlinear control systems based on differential geometry. A brief introduction about controllability and involutivity will be presented. As an example, the exact feedback linearization and the approximate feedback linearization are used in order to show some application examples.

KEY WORDS
Geometric Nonlinear Control, Exact Feedback Linearization, Approximate Feedback Linearization.

1. Introduction

The theory of nonlinear control systems have been receiving an increasing attention in the recent years, since it usually provides a better approach for real systems as well as better results than the linear approach. A lot of successful application examples can be found in strategic areas such as: aerospace, chemical and petrochemical industry, bioengineering and robotics.

"Nonlinear control systems" is a subject that deals with the analysis and the design of nonlinear control systems, i.e., the analysis and the design of control system that contains at least one nonlinear component [1]. The classic methods used in the study of the linear systems, particularly the frequency analysis, are not applicable to nonlinear systems [2]. Thus, different approaches are required in order to treat nonlinear systems. Even when considering a linear approach, if the required operation range gets larger, a linear controller may fail. It may become unstable or decrease the desired control performance. In these cases, techniques of nonlinear control may provide better performance. Examples of possible approaches may consider the system linearization around operation points and provide a gain schedule as well as a parameters adaptation (adaptive control). The exact feedback linearization and the approximate feedback linearization use a negative feedback signal that makes the closed loop system to have a linear behavior [1]. The present work consists of detailing with these two last techniques.

2. Using a Geometric Nonlinear Control to Analyse Necessary and Suficient Conditions To Feedback Linearization

The main objective of this paper is to provide an explicit and simple stabilizing controller for single-input single output nonlinear systems using the geometric nonlinear control for the exact feedback linearization and approximate feedback linearization.

Consider a standard smooth nonlinear control system affine in the input $u(t)$ given by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \qquad (1)$$

Where $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are C^∞ vector fields and $\dot{\mathbf{x}} \in R^n$. The system in the form (1) is said to be input-state linearizable if there exists a region Ω in R^n, such that the following conditions hold:

1. the matrix $\{\mathbf{g}, ad_\mathbf{f}\,\mathbf{g}, \ldots, \mathbf{ad}_\mathbf{f}^{n-1}\mathbf{g}\}$ has rank n,
2. the distribution

$$[ad_\mathbf{f}^i\,\mathbf{g}, ad_\mathbf{f}^j\,\mathbf{g}] = span\{\mathbf{g}, ad_\mathbf{f}\,\mathbf{g}, \ldots, ad_\mathbf{f}^{n-2}\,\mathbf{g}\} \qquad (2)$$

For $i, j = 0,1,2,.., n-2$

Where $[ad_\mathbf{f}^i\,\mathbf{g}, ad_\mathbf{f}^j\,\mathbf{g}]$ indicates Lie-Bracket, and $ad_\mathbf{f}^i\,\mathbf{g}$ is defined as [3]:

$$ad_\mathbf{f}^0\,\mathbf{g} = \mathbf{g} \text{ and } ad_\mathbf{f}^i\,\mathbf{g} = [\mathbf{g}, ad_\mathbf{f}^{i-i}\,\mathbf{g}] \qquad (3)$$

3. Exact Feedback Linearization

If the conditions (2) and (3) are satisfied, then it is possible find a scalar function $\phi(\mathbf{x})$ such that

$$\frac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}} ad_\mathbf{f}^i\,\mathbf{g} = 0 \quad i = 0,1,2,.., n-2 \qquad (4)$$

$$\frac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}} \, ad_{\mathbf{f}}^{n-1} \mathbf{g} \neq 0 \quad (5)$$

Where the $\phi(\mathbf{x})$, define a state transformation by

$$\mathbf{z} = T(\mathbf{x}) = [L_{\mathbf{f}}^{0}\phi(\mathbf{x}) \quad L_{\mathbf{f}}^{1}\phi(\mathbf{x}) \quad \ldots \quad L_{\mathbf{f}}^{n-1}\phi(\mathbf{x})] \quad (6)$$

An input control signal is than proposed as:

$$u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v = \frac{-L_{\mathbf{f}}^{n}\phi(\mathbf{x})}{L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\phi(\mathbf{x})} + \frac{1}{L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\phi(\mathbf{x})} v \quad (7)$$

Such that the closed-loop system in its new co-ordinates is described by a linear differential equation

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{b}\,v \quad (8)$$

Where

$$\mathbf{A} = \begin{bmatrix} 010 & \ldots 0 \\ 001 & \ldots 0 \\ \ldots\ldots\ldots & .. \\ 000 & \ldots 1 \\ 000 & \ldots 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix} \quad (9)$$

$\mathbf{A}$ and $\mathbf{b}$ are in the Brunovski canonical form. However, the generality is not lost since any representation of a linear controllable system is equivalent to the Brunovsky canonical form (4) through a state transformation.

3. Linearization of Pendulum System

Figure 1 gives a general view of the pendulum system, It considers a pendulum with a suspension held by a rigid connection rod (figure 1), being able to oscillate around this point. The pendulum is being excited by a force u .

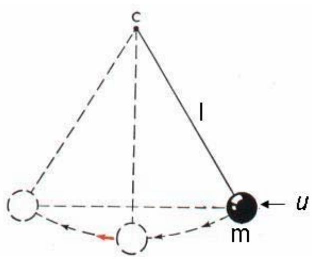

Figure 1. Pendulum

The model of this pendulum is given by:

$$\dot{x}_1 = x_2 \quad (10)$$

$$\dot{x}_2 = -\frac{g}{l}\operatorname{sen}(x_1) - \frac{b}{m}x_2 + \frac{1}{ml}u \quad (11)$$

Where g , l , m , b , x_1, x_2 are respectively the gravity acceleration, the length of the rod, the mass, the friction coefficients, the rotation angle and the angular speed. It is not difficult to verify that the conditions (1) and (2) are satisfied for this system. A suitable transfer function $T_1 = x_1$ can be found, and the coordinate and input the transformation are obtained by

$$T_2 = L_{\mathrm{f}}T_1 = \left[\frac{\partial T_1}{\partial x_1}\right]\begin{bmatrix} x_2 \\ \frac{g}{l}\operatorname{sen}(x_1) - \frac{b}{m}x_2 \end{bmatrix} = \\ = [1 \quad 0]\begin{bmatrix} x_2 \\ \frac{g}{l}\operatorname{sen}(x_1) - \frac{b}{m}x_2 \end{bmatrix} = x_2 \quad (12)$$

$$\alpha(\mathbf{x}) = \frac{-\frac{g}{l}\operatorname{sen}(x_1) - \frac{b}{m}x_2}{\frac{1}{ml}} = mg\operatorname{sen}(x_1) + lbx_2 \quad (13)$$

$$\beta(\mathbf{x}) = -\frac{1}{\frac{1}{ml}} = -ml \quad (14)$$

Then, a set of linear equations is achieved in order to represent the closed loop system.

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{bmatrix} = \begin{bmatrix} T_2 \\ -v \end{bmatrix} \quad (15)$$

The block diagram of the system is represented as

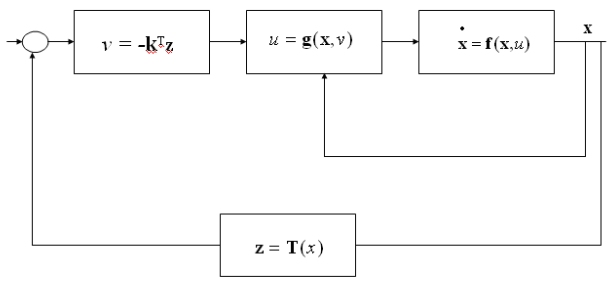

Figure 2. Controller Structure

The design's concept is reflected by the two-loop structure of the controller. In the first step, the design seeks nonlinear compensation, which explicitly eliminates the nonlinearities present in the system. The second step is the design of a linear controller to the system.

4. Approximate Feedback Linearization

In this section, a higher order approximate linearization is considered and the technique is then proposed to be applied to the inverted pendulum system. Notice that this system cannot be exactly linearizable since the model of the inverted pendulum fails to satisfy the involutivity condition [4], [5], [6]. Thus, an approximate linearization approach is proposed.

The inverted pendulum control, Figure 2, is a classical educational problem which consists in a balancing pole built on the top of a one-dimensional movable cart. [5], [6].

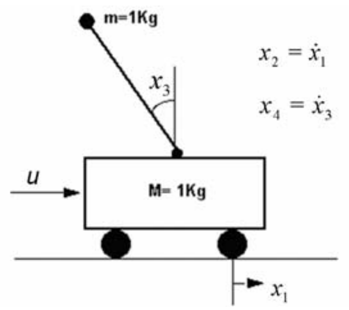

Figure 3.Inverted Pendulum

Consider the following single-input nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \qquad (16)$$

Letting $x_1 = x$, $x_2 = \dot{x}_1$, $x_3 = \theta$, $x_4 = \dot{\theta}$, the equations of motion can be written as [6]

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ \dfrac{\sin(x_3)x_4^2 - g\sin(x_3)\cos(x_3)}{2-\cos(x_3)^2} \\ x_4 \\ \dfrac{-\sin(x_3)\cos(x_3)x_4^2 + 2g\sin(x_3)}{2-\cos(x_3)^2} \end{bmatrix} \qquad (17)$$

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ \dfrac{1}{2-\cos(x_3)^2} \\ 0 \\ \dfrac{-\cos(x_3)}{2-\cos(x_3)^2} \end{bmatrix} \qquad (18)$$

For order 1 involutivity, there must exist coefficients $c_{ij}(x_0)$
such that constant term
$\mathbf{e}_{ij}(\mathbf{x}) = \mathbf{e}_{ij}(\mathbf{x_0}) + \sum_{k=1}^{n} \dfrac{\partial \mathbf{e}_{ijk}(\mathbf{x_0})}{\partial \mathbf{x}_k}\tilde{\mathbf{x}}_k + O^2(\tilde{\mathbf{x}})$ is annihilated [7], i.e.

$$\mathrm{x}_{ij}(\mathrm{x}_0) - \mathrm{D}_I(\mathrm{x}_0)\mathrm{c}_{ij}(\mathrm{x}_0) = 0 \qquad (19)$$

Where

$$\mathbf{x}_{ij} = [\mathbf{x}_i, \mathbf{x}_j](\mathbf{x}) \qquad (20)$$

And

$$D_I = \{\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}, ..., ad_{\mathbf{f}}^{\mathbf{n-2}}\,\mathbf{g}\} \qquad (21)$$

Note that this is a linear problem. Thus, this system is order 1 involutivity. There are no solutions for order 2 involutivity.

As the system is order 1 involutive, then λ_1 can be obtained by solving

$$\nabla\lambda_1 \mathrm{D}_0 = 0 \qquad (22)$$

Obtaining

$$\nabla\lambda_1 = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^T \qquad (23)$$

Once λ_1 has been computed, the remaining states and control transformations are obtained from [8]

$$\begin{aligned} \frac{\partial}{\partial t}\begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n\end{bmatrix} &= L_{\mathrm{f}+\mathrm{g}u}\begin{bmatrix}\lambda_1 \\ \vdots \\ L_{\mathrm{f}}^{\mathrm{n}\text{-}2}\lambda_1 \\ L_{\mathrm{f}}^{\mathrm{n}\text{-}1}\lambda_1\end{bmatrix} = \\ &= \begin{bmatrix}\lambda_2 \\ \vdots \\ \lambda_n \\ L_{\mathrm{f}}\lambda\end{bmatrix} + \begin{bmatrix}L_{\mathrm{g}}\lambda_1 \\ \vdots \\ L_{\mathrm{g}}L_{\mathrm{f}}^{n\text{-}2}\lambda_1 \\ L_{\mathrm{g}}L_{\mathrm{f}}^{n\text{-}1}\lambda_1\end{bmatrix}u \end{aligned} \qquad (24)$$

The last column in the right hand side of (24) represents the neglected terms in the transformed system.

The equation (24) put the input control transformation as

$$u = \alpha(\mathrm{x}) + \beta(\mathrm{x})v \qquad (25)$$

Where,

$$\alpha(\mathrm{x}) = \frac{-L_{\mathrm{f}}^{n}\lambda_1}{L_g L_{\mathrm{f}}^{n\text{-}1}\lambda_1} \text{ and } \beta(\mathrm{x}) = \frac{1}{L_g L_{\mathrm{f}}^{n\text{-}1}\lambda_1} \qquad (26)$$

They lead to an approximated linearized system as

$$\frac{\partial}{\partial t}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{n-1}\\ \lambda_n\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{n-1}\\ \lambda_n\end{bmatrix}+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}v \quad (27)$$

Where,

$$\dot{\lambda}_i = \lambda_{i+1} + O^{p+1}(\tilde{\mathbf{x}},u) \quad (28)$$

$$\dot{\lambda}_n = v + O^{p+1}(\tilde{\mathbf{x}},u) \quad (29)$$

5. Conclusion

Here we have designed a nonlinear controller for a pendulum systems based on the exact feedback linearization and the inverted pendulum systems based on the approximate linearization technique using a differential geometric to calculate the state transformation and input transformation.

References

[1] J.J. Slotine, L. Weiping, *Applied nonlinear control*. (Prentice-Hall, 1991).

[2] M. Ondera, Matlab-based tools for nonlinear systems.

[3] J. Bronislaw. Introduction to geometric nonlinear control; controllability and lie bracket, September 2001.

[4] G. Guadarbassi, S. Savaresi, Approximate linearization via feedback – an overview, *Automatica 2001*.

[5] S. Renou, S. Stéphane, Real time control of an inverted pendulum based on approximate linearization. *Ieee 1993.*

[6] J. Deutscher, C. Schmid, A state space embedding approach to approximate feedback linearization of single input nonlinear control systems, *International Journal Robust Nonlinear Control* 2005.

[7] L. Guzzela; A Isidori. On approximate linearization of nonlinear control systems. *International Journal Of Robust And Nonlinear Control, Vol.3*, 1993, 261-276.

[8] N. Bedrossian. Approximate feedback linearization: the cart-pole example.

[9] A. J. Krener, approximate linearization by state feedback and coordinate change. *Systems And Control Letters*, 1984, 181-185.